Classification of unitary R-matrices by braid group characters

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1. The classification programme for unitary R-matrices

The structure of the set of the solutions to the Yang-Baxter Equation (YBE) remains mysterious despite its connections to many fields in mathematics and its manifold applications. Classification results on all solutions, or some large subset of solutions, seem hard to come by unless one considers them up to a suitable equivalence relation.

Since applications in quantum physics often require *unitary* solutions, we consider here pairs (R, V) where V is a finite-dimensional Hilbert space and $R : V \otimes V \to V \otimes V$ a unitary endomorphism satisfying the YBE, namely

$$(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R).$$

We denote the collection of all such "R-matrices" by $\mathcal{R}(V)$ and propose to consider it up to a natural equivalence relation related to braid group characters.

Recall that any $R \in \mathcal{R}(V)$ defines unitary representations $\rho_R^{(n)}$ of the braid groups B_n, $n \in \mathbb{N}$, on the tensor powers $V^{\otimes n}$, by mapping the i-th Artin generator σ_i to $R_i = \mathrm{id}_V^{\otimes(i-1)} \otimes R \otimes \mathrm{id}_V^{\otimes(n-i-1)}$.

Definition 1.1. [2] Two unitary R-matrices R, S are called equivalent, written $R \sim S$, if for all $n \in \mathbb{N}$ their B_n-representations are unitarily equivalent.

An equivalent way of expressing $R \sim S$ is to say that R and S are defined on Hilbert spaces of the same dimension and define the same normalized character $\chi_R = \chi_S$ of the infinite braid group B_∞ [3].

The above equivalence leads to large equivalence classes and hence to the much more accessible (but still challenging) problem of classifying $\mathcal{R}(V)$ up to $\sim$. In the special case of *involutive* (meaning $R^2 = 1$) unitary solutions, a complete classification has been achieved. The equivalence classes of involutive solutions are in explicit bijection with pairs of Young diagrams with $\dim V$ cells in total [2]. Also all unitary Temperley-Lieb solutions of the YBE have been classified, here the classifying data are the spectrum of R and the value a spectral projection takes in a Markov trace [1].

In general, the classification problem for $\mathcal{R}(V)/\sim$ is open. This is partly due to the fact that only few accessible invariants for $\sim$ are known, namely a) the characteristic polynomial of R, capturing all character values of the form $\chi_R(\sigma_1^k)$, $k \in \mathbb{Z}$ and b) the characteristic polynomial of the partial trace[1] of R, capturing all character values of cycle form, namely

$$\chi_R(\sigma_1\sigma_2\cdots\sigma_{n-1}) = (\dim V)^{-n} \cdot \mathrm{Tr}_V((\mathrm{ptr}R)^{n-1}).$$

[1]Recall that the partial trace is the map $\mathrm{ptr} : \mathrm{End}(V \otimes V) \to \mathrm{End}(V)$, $\mathrm{ptr} = \mathrm{Tr}_V \otimes \mathrm{id}_{\mathrm{End}(V)}$.

2. Racks, Quandles and Set-Theoretic Solutions to the YBE

Let X be a finite set and $r : X \times X \to X \times X$ a *set-theoretic* bijective, non-degenerate solution ("set-theoretic solution", for short) to the YBE. Then its linearisation, namely the Hilbert space $V = \mathrm{span}(X)$ spanned by X as an orthonormal basis, and the linear extension R of r, yields a unitary solution to the linear YBE.

The classification of the subset $\mathcal{R}_{\mathrm{set}}(V) \subset \mathcal{R}(V)$ coming from non-degenerate set-theoretic solutions up to $\sim$ is currently being investigated in the PhD project of the first named author. We here report on some observations and results in this project.

Recall that invertible maps $r : X^2 \to X^2$ of the special form

$$r(x, y) = (\lambda_x(y), x), \qquad x, y \in X,$$

with $\lambda_x \in S_X$ bijections, solve the YBE if and only if $x \triangleright y := \lambda_x(y)$ is a rack structure on X. We will refer to such solutions as *rack solutions* for short.

Proposition 2.1. [4, 5] *Any $R \in \mathcal{R}_{\mathrm{set}}(V)$ is equivalent to a rack solution.*

This result tells us that up to equivalence, we may restrict attention from $\mathcal{R}_{\mathrm{set}}(V)$ to rack solutions. As the only involutive rack solution is the flip $r(x, y) = (y, x)$, we also have:

Corollary 2.2. *All non-degenerate set-theoretic involutive solutions on X are equivalent.*

Comparing with the results of [2], we see that this is due to the assumption of non-degeneracy – in fact, many of the normal form R-matrices in [2] are set-theoretic (but degenerate).

Isomorphic racks $(X, \triangleright) \cong (X', \triangleright')$ give rise to equivalent R-matrices, since the isomorphism produces a permutation matrix intertwiner for the R-matrices. Given the much larger flexibility of equivalence $\sim$ in comparison to rack isomorphism, it is not expected that equivalence of two rack solutions implies isomorphism of their underlying racks. Nonetheless, examples of non-isomorphic equivalent racks are non-trivial to find. The main question in this context is to understand how much of the structure of a rack $(X, \triangleright)$ is encoded in the B_∞-character of its associated rack solution r.

The two invariants mentioned in the first section translate as follows. The characteristic polynomial of the linearisation R of r exactly encodes the conjugacy class of r in S_{X^2}. It is currently not clear to us which conjugacy classes in S_{X^2} are realized by rack structures on X.

To translate the partial trace of R, we first recall that the *square map* of a rack $(X, \triangleright)$ is the map $\mathrm{Sq} : X \to X$, $\mathrm{Sq}(x) := x \triangleright x$. The square map of a rack is always a bijection of X. Any $\pi \in S_X$ arises as the square map of some rack, for instance from the permutation solution example $x \triangleright y = \pi(y)$, $x, y \in X$.

Lemma 2.3. *The partial trace of the linearisation of a rack solution r is the linearisation of the square map of the underlying rack.*

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$N(d)$	1	0	1	1	3	2	5	3	8	1	9	10	11	0
$E_{23}(d)$	–	–	–	–	1	1	1	0	4	–	6	4	2	–

FIGURE 1. $d = |X|$, $N(d)$: number of isomorphism classes of indecomposable quandles of size d, $E_{23}(d)$: number of pairs of classes with equivalent B_2- and B_3-actions on X^2 and X^3.

Hence the characteristic polynomial of the partial trace of the linearisation of a rack solution exactly encodes the conjugacy class of the square map in S_X, and all conjugacy classes are realized.

However, these two conjugacy classes (of r in S_{X^2} and of Sq in S_X) appear to be insufficient to determine the equivalence class of r up to $\sim$. For instance, when restricting to quandles (i.e. racks with $x \triangleright x = x$ for all $x \in X$) the square map is always trivial, but many non-equivalent quandles ought to exist.

To shed some light on this question, we did a numerical investigation of small indecomposable quandles provided in the gap package rig [6]. Searching for quandles with equivalent B_2- and B_3-representations (with set-theoretic intertwiners), we found only few pairs of candidates for equivalent solutions. For example, there exists a single pair out of the 10 possible ones for $|X| = 7$, and no pair at all out of the three possible ones for $|X| = 8$ (see Figure 1). It is currently not yet decided whether these pairs have equivalent B_n-representations for arbitrary numbers of strands n.

We expect to find examples of non-isomorphic but equivalent quandles within these data, for instance by considering Alexander quandles.

ACKNOWLEDGEMENTS

GL would like to thank Victoria Lebed for illuminating discussions on quandles and Leandro Vendramin for further discussions and advice on rig/gap. Financial support from the EPSRC and the DFG through the Heisenberg professorship "Quantum Fields & Operator Algebras" (LE 2222/3-1) is gratefully acknowledged.

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