Yang-Baxter Representations of the Infinite Symmetric Group

Gandalf Lechner

joint work with Ulrich Pennig and Simon Wood

The Yang-Baxter equation and the infinite symmetric group

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1
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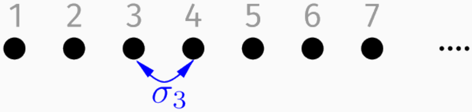
6
7
....
σ_2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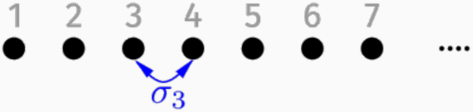
σ_3

The Yang-Baxter equation and the infinite symmetric group

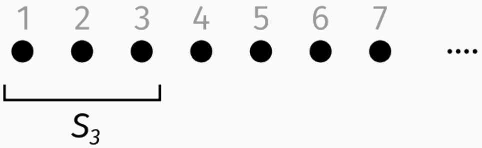


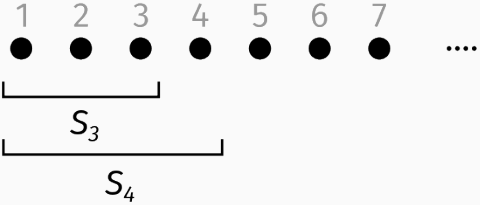
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

$$\sigma_i^2 = e$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \ |i - j| > 1$$

The Yang-Baxter equation and the infinite symmetric group




1 2 3 4 5 6 7
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S_3
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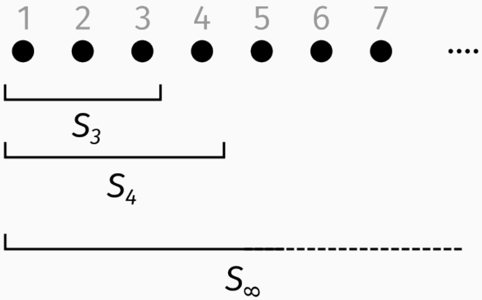

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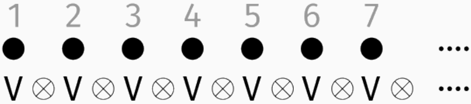

The Yang-Baxter equation and the infinite symmetric group

1 2 3 4 5 6 7

$$V \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \otimes \cdots$$

$$R_1 = \boxed{R} \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \cdots$$

$$R \in \mathrm{End}(V \otimes V)$$

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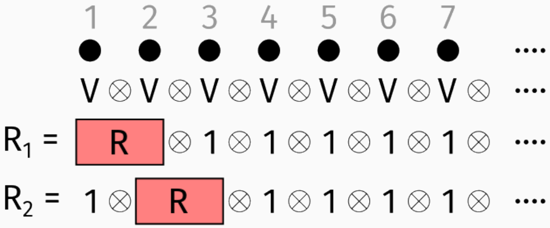

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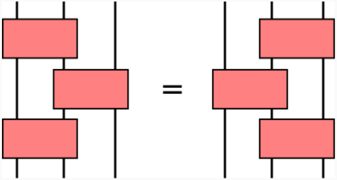
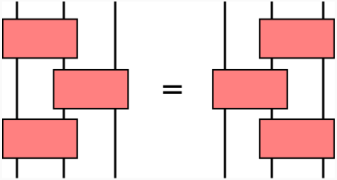
Yang-Baxter equation:

$$R_1 R_2 R_1 = R_2 R_1 R_2.$$

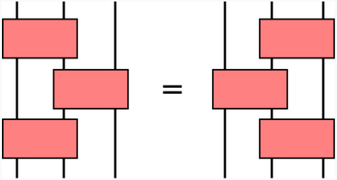
The Yang-Baxter equation and the infinite symmetric group





Definition (for purpose of this talk)

V : finite-dim. Hilbert space. An **R-matrix** is a unitary $R \in \mathrm{End}(V \otimes V)$ such that $R_1 R_2 R_1 = R_2 R_1 R_2$ and $R^2 = 1$.



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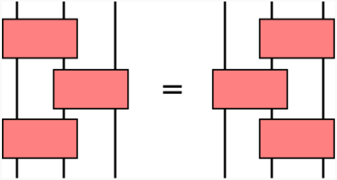

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- $\mathcal{R}_0 :=$ set of all R-matrices (with any V)
- Any $R \in \mathcal{R}_0$ gives unitary rep. $\rho_R^{(n)}$ of S_n on $V^{\otimes n}$ via

$$\rho_R^{(n)}(\sigma_i) := R_i, \qquad i = 1, \ldots, n-1$$

$$\rho_R : S_\infty \to \bigotimes_{n \geq 1} \mathrm{End}\, V$$

Motivated from QFT constructions [Alazzawi-GL 2016]:

Definition

$R, S \in \mathcal{R}_0$ are called equivalent,

$$R \sim S,$$

if for each n, the S_n-representations $\rho_R^{(n)} \cong \rho_S^{(n)}$ are equivalent.

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Simple observations:

- $R \sim S \Longrightarrow \dim R = \dim S, \operatorname{Tr} R = \operatorname{Tr} S$.
- For each $A \in \mathrm{GL}(V)$,

$$R \sim (A \otimes A) R (A^{-1} \otimes A^{-1})$$
$$R \sim FRF$$

Question 1

Find *all* R-matrices up to equivalence: Find a parameterization of $\mathcal{R}_0/\sim$ and a representative in each equivalence class.

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Question 3

Which reps ρ of S_∞ are of the form $\rho \cong \rho_R$ for some $R \in \mathcal{R}_0$? ("Yang-Baxter representations")

Yang-Baxter characters of S_∞

Normalized trace on tensor products ($d = \dim V$):

$$\tau = \frac{\mathrm{Tr}_V}{d} \otimes \frac{\mathrm{Tr}_V}{d} \otimes \frac{\mathrm{Tr}_V}{d} \otimes \ldots$$

For each R,

$$\boldsymbol{\chi_R := \tau \circ \rho_R} : S_\infty \longrightarrow \mathbb{C}$$

is a (normalized) character of S_∞.

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- normalized character of S_∞ = tracial state on $C^* S_\infty$
- On n-cycle $c_n : i_1 \mapsto i_2 \mapsto \ldots \mapsto i_n \mapsto i_1$, get

$$\chi_R(c_n) = d^{-n}\ \mathrm{Tr}_{V^{\otimes n}}(R_1 \cdots R_{n-1}).$$

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- χ_R "factorizes": For $\sigma, \sigma' \in S_\infty$ with disjoint supports,

$$\chi_R(\sigma\sigma') = \chi_R(\sigma) \cdot \chi_R(\sigma')\,.$$

Theorem [Thoma '64]

(1) A character χ of S_∞ is extremal if and only if it factorizes.

(2) $\mathbb{T} :=$ all real sequences $\{\alpha_i\}_i, \{\beta_i\}_i$ such that

- $\alpha_i \geq \alpha_{i+1} \geq 0$, $\beta_i \geq \beta_{i+1} \geq 0$
- $\sum_i (\alpha_i + \beta_i) \leq 1$

Extremal characters are in 1 : 1 correspondence with $\mathbb{T}$ via

$$\chi(c_n) = \sum_i \alpha_i^n + (-1)^{n+1} \sum_i \beta_i^n, \qquad n \geq 2.$$

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- Which Thoma parameters are realized by Yang-Baxter characters?

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Theorem [Wassermann '81]

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- Thus: Thoma parameters (α, β) of a YB character satisfy $\sum_i(\alpha_i + \beta_i) = 1$, and only finitely many are non-zero.

Yang-Baxter subfactors

Notation:

$$\mathcal{E} := \overline{\bigotimes_{n\geq 1} \operatorname{End} V}^{\tau}$$

$$\mathcal{M}_R := \rho_R(S_\infty)'' = \{R_i \,:\, i \in \mathbb{N}\}'' \subset \mathcal{E}$$

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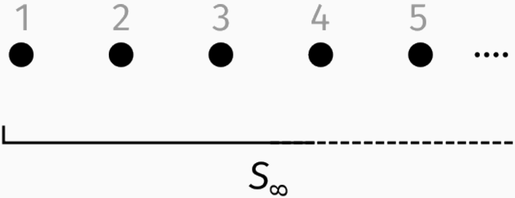
- $\mathcal{N}_R' \cap \mathcal{M}_R = \mathbb{C}$ if and only if $R \in \{\pm 1, \pm F\}$ [Gohm-Köstler 2010, Yamashita 2012]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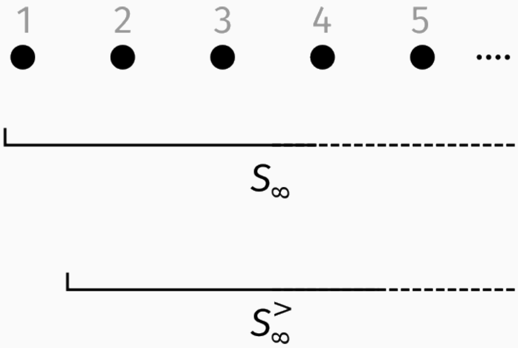
S∞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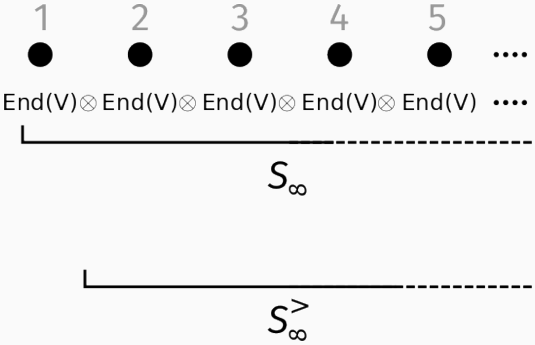
$S_\infty^>$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S_∞
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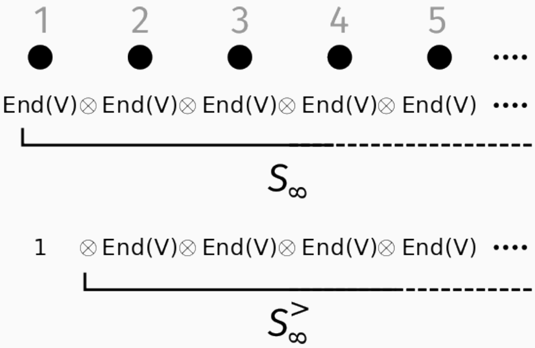
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Compare the subfactors

$$\mathcal{N}_R \subset \mathcal{M}_R, \qquad \mathcal{N}_R' \cap \mathcal{M}_R \subset \mathcal{M}_R$$

to tensor product subfactors

$$1 \otimes \operatorname{End} V \otimes \operatorname{End} V \otimes \ldots \subset \mathcal{E} \qquad \operatorname{End} V = \operatorname{End} V \otimes 1 \otimes 1 \ldots \subset \mathcal{E}.$$

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In both cases, have τ-preserving conditional expectations:

- $\operatorname{End} V \subset \mathcal{E}$: Cond. exp. $E =$ partial trace

$$E : \mathcal{E} \longrightarrow \operatorname{End} V, \quad E = \mathrm{id}_{\operatorname{End} V} \otimes \tau \otimes \tau \otimes \ldots$$

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$$\begin{array}{ccc} \operatorname{End} V & \xleftarrow{\;E\;} & \mathcal{E} \\ & & \uparrow \\ \mathcal{N}_R' \cap \mathcal{M}_R & \xleftarrow{\;E_R\;} & \mathcal{M}_R \end{array}$$

Proposition

$E(R_1) = E_R(R_1)$.

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With arguments from [Gohm-Köstler 2010], one then gets

Theorem

Let $c_n \in S_\infty$ be an n-cycle, $n \geq 2$. Then

$$\chi_R(c_n) = \tau(E(R_1)^{n-1})$$

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Theorem: Characterization of $\sim$

Define the "usual partial trace" of R as

$$\operatorname{ptr} R := (\mathrm{id}_{\operatorname{End} V} \otimes \operatorname{Tr}_V)(R).$$

$$\Rightarrow \chi_R(c_n) = d^{-n} \operatorname{Tr}_V(\operatorname{ptr}(R)^{n-1}).$$

$R \sim S$ if and only if $\mathbf{ptr}R \cong \mathbf{ptr}S$.

partial trace in $d = 2$:

$$\begin{pmatrix} a & b & a' & b' \\ c & d & c' & d' \\ a'' & b'' & a''' & b''' \\ c'' & d'' & c''' & d''' \end{pmatrix}$$

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spectrum of partial trace of R determines equivalence class $[R]$.

spectral characterizations also appear in [Okounkov 99]

Write

$$\chi_R(c_n) = d^{-n}\,\mathrm{Tr}_V(\mathrm{ptr}(R)^{n-1})$$

in Thoma parameters (α, β) of R and eigenvalues t_j of $\mathrm{ptr}\,R$:

$$\sum_i \alpha_i^n + (-1)^{n+1}\sum_i \beta_i^n = d^{-n}\sum_j t_j^{n-1}\,.$$

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This implies:

Corollary

The Thoma parameters of a YB character are **rational**.

Normal form R-matrices

So far:

(1) $R \sim S$ if and only if $\operatorname{ptr} R \cong \operatorname{ptr} S$.

(2) Thoma parameters of YB characters lie in $\mathbb{T}_{\mathrm{YB}} \subset \mathbb{T}$, defined by:

- Only finitely many α_i, β_i are non-zero
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Now:

- Given $(\alpha, \beta) \in \mathbb{T}_{\mathrm{YB}}$, construct R with these parameters.

Normal form R-matrices

So far:

(1) $R \sim S$ if and only if $\operatorname{ptr} R \cong \operatorname{ptr} S$.

(2) Thoma parameters of YB characters lie in $\mathbb{T}_{\mathrm{YB}} \subset \mathbb{T}$, defined by:

- Only finitely many α_i, β_i are non-zero
- $\sum_i(\alpha_i + \beta_i) = 1$
- $\alpha_i, \beta_i \in \mathbb{Q}$

Now:

- Given $(\alpha, \beta) \in \mathbb{T}_{\mathrm{YB}}$, construct R with these parameters.
- Plan: Build R-matrix from simple blocks by “direct sum”

Setting: V, W Hilbert spaces, $X \in \mathrm{End}(V \otimes V)$, $Y \in \mathrm{End}(W \otimes W)$.
Define

$$X \boxplus Y \in \mathrm{End}((V \oplus W) \otimes (V \oplus W))$$

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$$(V \oplus W) \otimes (V \oplus W) = (V \otimes V) \oplus (W \otimes W) \oplus ((V \otimes W) \oplus (W \otimes V)).$$

[Lyubashenko 87, Gurevich 91, Hietarinta 93]

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Proposition

- $\boxplus$ is commutative and associative.
- $\boxplus$ preserves the YBE: $R, S \in \mathcal{R}_0 \Rightarrow R \boxplus S \in \mathcal{R}_0$.
- $\mathrm{ptr}(R \boxplus S) = \mathrm{ptr}\, R \oplus \mathrm{ptr}\, S$.

Let $d_1^+, \ldots, d_n^+, d_1^-, \ldots, d_m^- \in \mathbb{N}$. Normal form R-matrix (with dimensions d^+, d^-) is defined as

$$N := 1_{d_1^+} \boxplus \ldots \boxplus 1_{d_n^+} \boxplus (-1_{d_1^-}) \boxplus \ldots \boxplus (-1_{d_m^-}).$$

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Theorem

- *Let $d := d_1 + \ldots + d_n$. Then χ_N has Thoma parameters*

$$\alpha_i = \frac{d_i^+}{d}, \qquad \beta_j = \frac{d_j^-}{d}.$$

- *Yang-Baxter characters are in 1:1 correspondence with $\mathbb{T}_{\mathrm{YB}}$.*

$\mathcal{R}_0/\sim \;\cong \mathbb{Y} \times \mathbb{Y}$

It is convenient to rescale the Thoma parameters by the dimension:

$$a_i := d\alpha_i, \qquad b_i := d\beta_i\,.$$

These are **integers** ($= |$eigenvalues of ptr $R|$), and sum to d.

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Theorem

$\mathcal{R}_0/\sim$ is in 1:1 correspondence with $\mathbb{Y} \times \mathbb{Y}$ via

$$[R] \mapsto (a, b)$$

- Example:

$$\left(\yng(3,1), \yng(2,2)\right) : \qquad d = 8, \alpha = \left(\frac{3}{8}, \frac{1}{8}\right), \beta = \left(\frac{1}{4}, \frac{1}{4}\right).$$

Algebraic structure of $\mathcal{R}_0/{\sim}$:

- "sum $\boxplus$"

$$R = (3+2, 5+1+1), \quad S = (2, 7+1)$$
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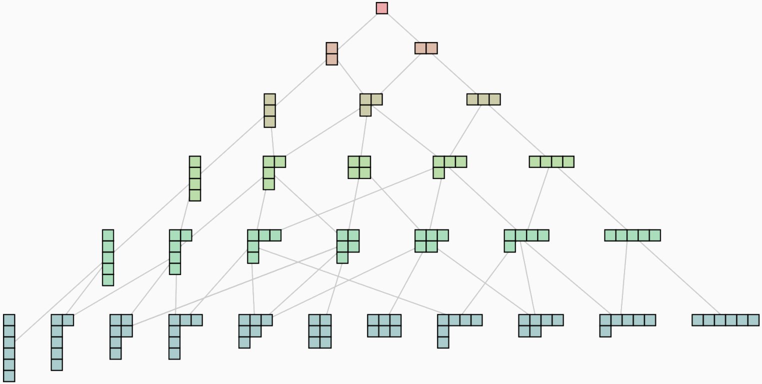
- $\mathcal{R}_0/\sim$ has structure of a "rig" (ring without negatives).

Describe the multiplicities of the irreps of S_n in $\rho_R^{(n)}$.



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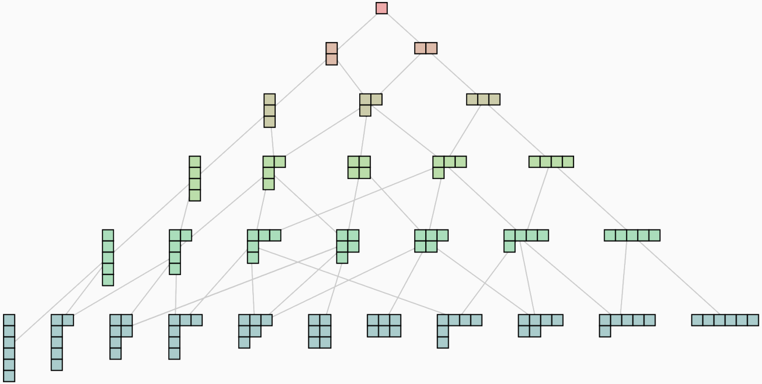

- $B(R)$:= $(1 + \#$ non-zero α's$) \times (1 + \#$ non-zero β's$)$
- For example: $B(R) =$  for $\alpha = 0$, $\beta = (\frac{2}{3}, \frac{1}{3})$

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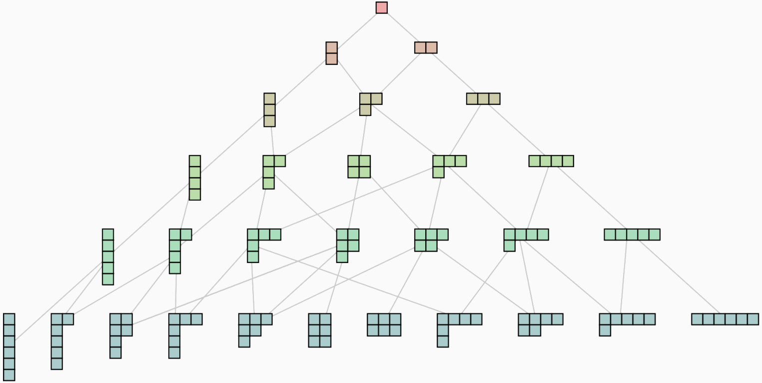

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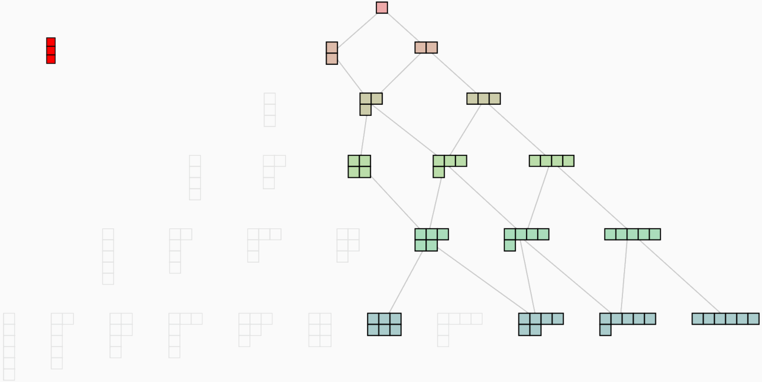

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Outlook

The following generalizations are on our agenda:

- Introduce a spectral parameter $\longrightarrow$ QFT!
- Drop the assumption $R^2 = 1 \longrightarrow$ braid groups!