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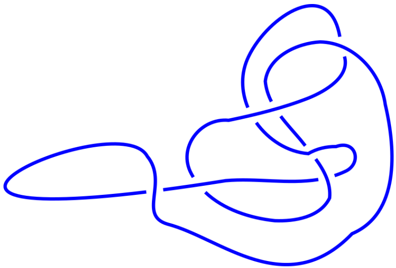
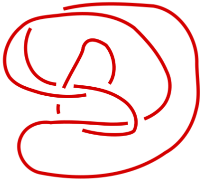


J



K

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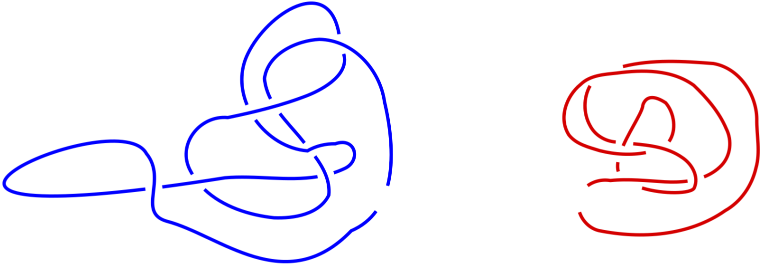

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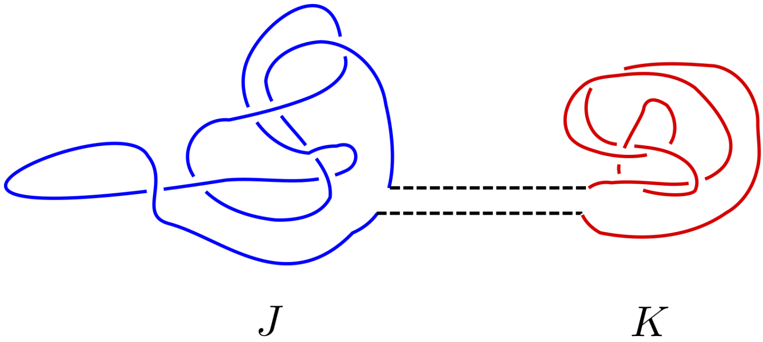

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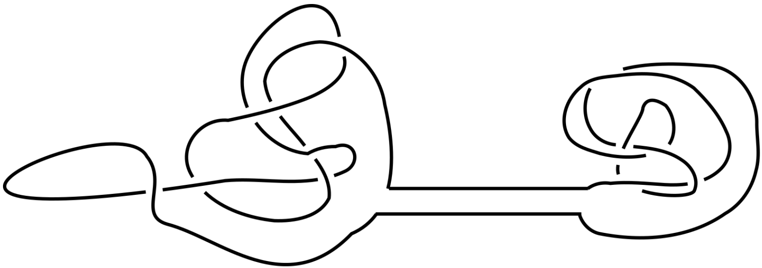

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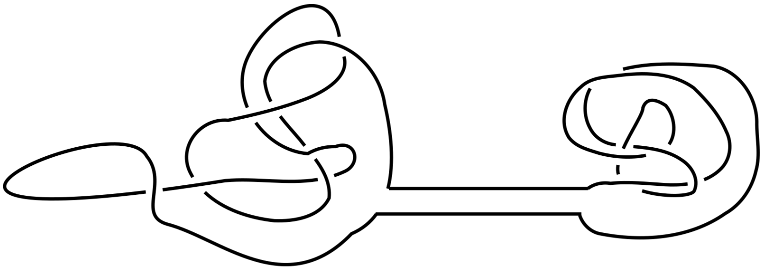

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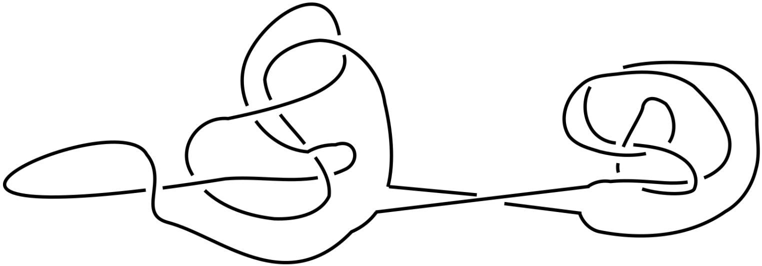

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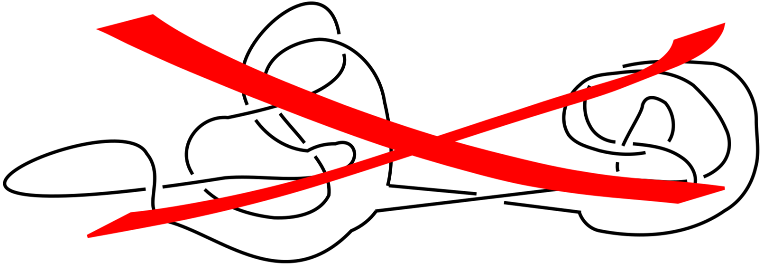

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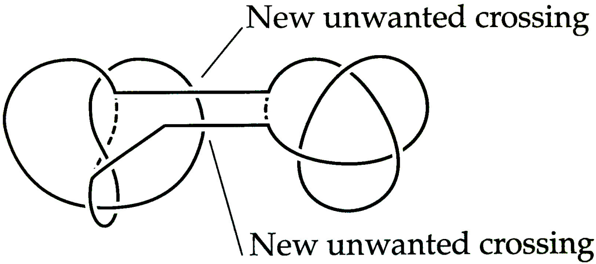

Figure 1.11 *Not* the composition of *J* and *K*.

Picture taken from Adams' book

Question: Is the composition $\mathcal{J}\#K$ *well-defined*, i.e. independent of the choices of diagrams for $\mathcal{J}$, K, and the choice where the knots are opened?

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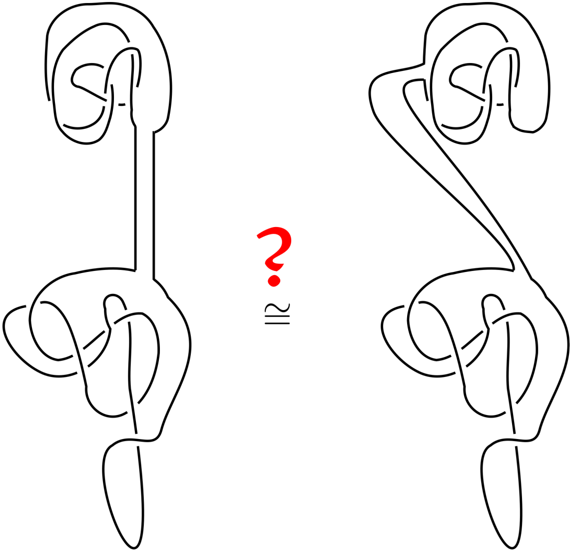

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But the composition **is** well-defined if $\mathcal{J}, K$ are **oriented**, and the orientations match up in $\mathcal{J}\#K$.

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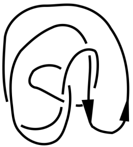

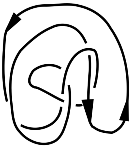

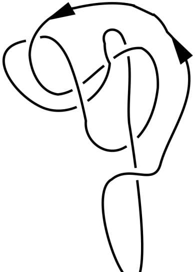

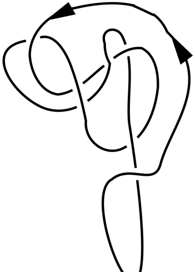

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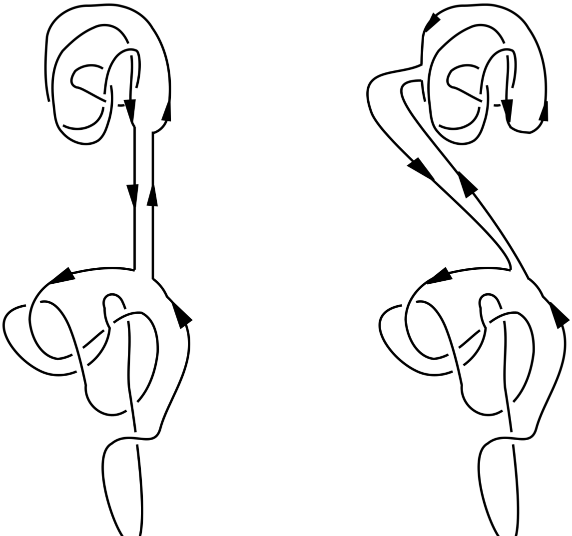

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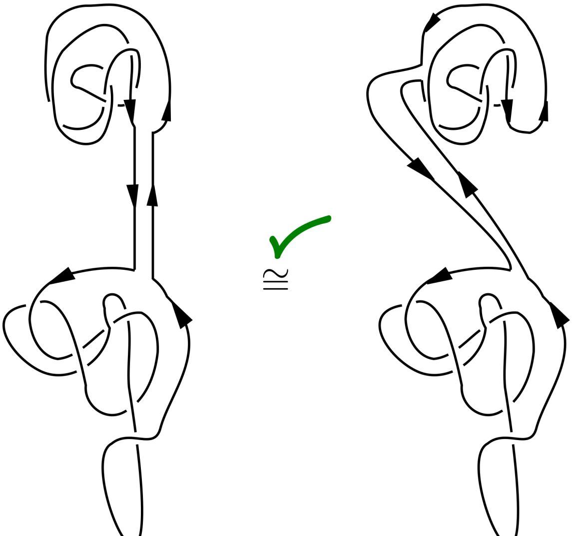

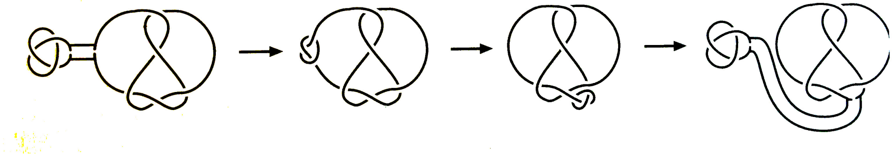

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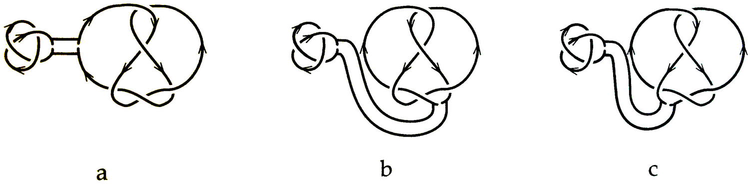
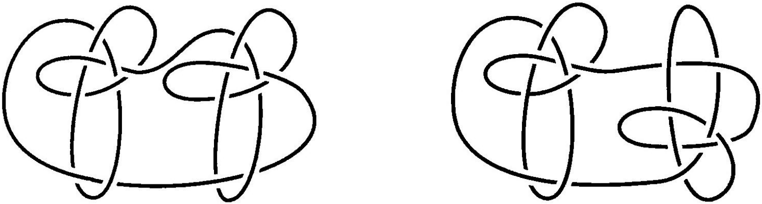


Figure 1.17 (a) Orientations match. (b) Orientations match. (c) Orientations differ.

Picture taken from Adams' book



$$8_{17}\#8_{17} \neq 8_{17}\#m(8_{17})$$

Picture taken from Adams' book

Theorem 3.10

- The composition $\mathcal{J}\#\mathcal{K}$ of oriented topological knots $\mathcal{J}, \mathcal{K}$ is well-defined.
- If $\mathcal{K}$ is any oriented topological knot, and $\mathcal{U}$ the oriented topological unknot, then

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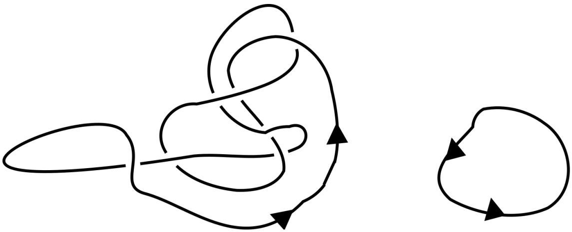

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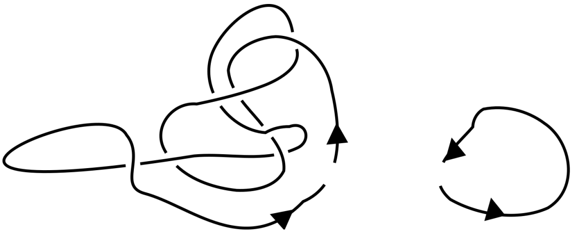

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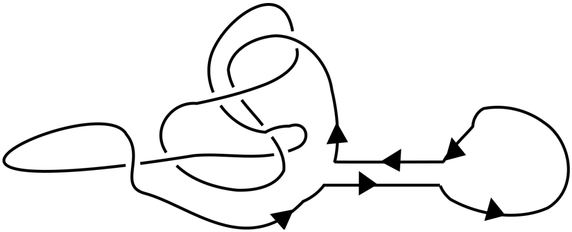

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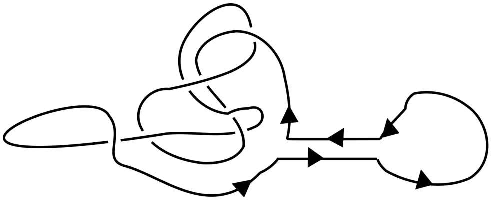

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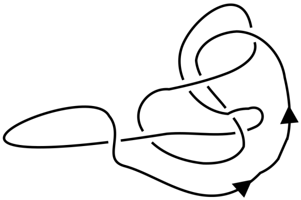


Definition 3.11

A topological knot $\mathcal{K}$ which can be written as $\mathcal{K} = \mathcal{K}_1 \# \mathcal{K}_2$, with both $\mathcal{K}_1$ and $\mathcal{K}_2$ non-trivial, is called *composite*.

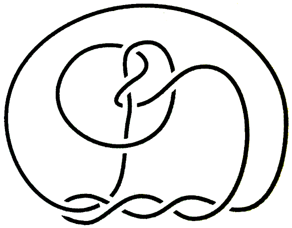
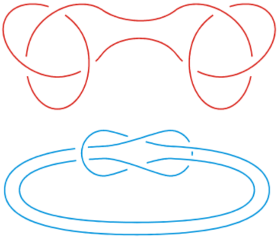


Figure 1.15 A composite knot.

Picture taken from Adams' book

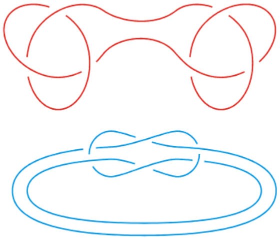
Example:

trefoil # mirror trefoil



reef knot

trefoil # trefoil



granny knot

Analogy to integers and primes:

integers	**knots**
multiplication of integers $n \cdot m$	composition of knots $\mathcal{J}\#\mathcal{K}$
1	unknot
composite integer $n = m \cdot k$, $m, k \neq 1$	composite knot

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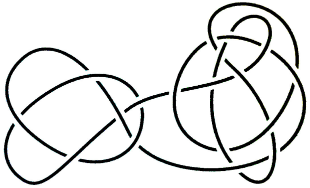

Figure 1.14 Could this untangle to be the unknot?

No. Like $1 \neq m \cdot k$ for integers $m, k \neq 1$, the unknot is not composite.

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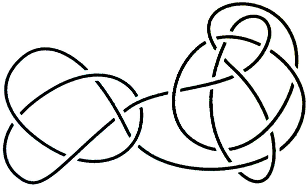

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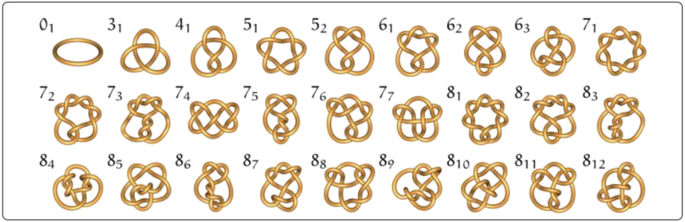
The analogy goes even further:

integers	**knots**
1 is not composite	The unknot is not composite
Define a **prime number** to be a number that is not composite, and not 1	Define a **prime knot** to be a knot that is not composite, and not the unknot
Each integer n factors into a unique set of prime numbers $n = p_1 \cdots p_N$	Each knot $\mathcal{K}$ factors into a unique set of prime knots $\mathcal{K} = \mathcal{P}_1 \# \ldots \# \mathcal{P}_N$

Knot Catalogues

- Knot catalogues list prime knots (and the unknot), like a list of prime numbers (and 1): $1, 2, 3, 5, 7, 11, 13, 17, \ldots$



- Notation: n_m means knot "number m" (in an arbitrary order) with crossing number n.
- Note: Mirror images are not listed, although they can be inequivalent (as in the case of the trefoil $3_1 \neq 3_1^*$, for example)

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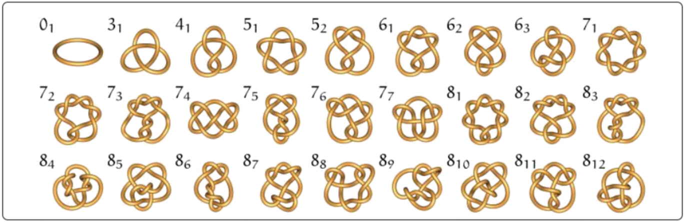

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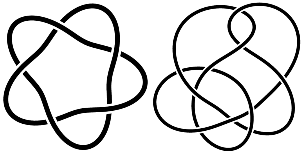

$$V_{5_1}(t) = V_{10_{132}}(t) = -t^{-7} + t^{-6} - t^{-5} + t^{-4} + t^{-2}$$

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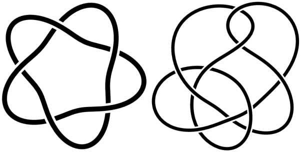

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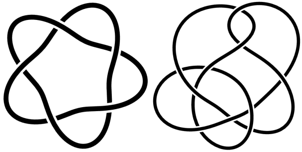

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- Number of knots with crossing number up to 10 that are different from their mirror image (chiral): **229**.
- Up to two exceptions, 9_{42} and 10_{125}, the Jones polynomial detects these chiralities: $V(K) \neq V(K^*)$ unless $K = 9_{42}$ or $K = 10_{125}$.

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Answer: Unknown.