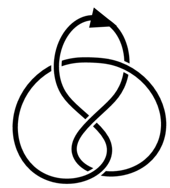
Knots – Exercise Sheet 5

Module MA0322 – Gandalf Lechner – Autumn 2015

handed out: 24 November
solutions can be handed in until: 3 December
will be discussed in class: 4 December

Problem 14: Partition functions

Consider the diagram D of the oriented figure of eight knot on the right, an arbitrary index set I, and tensors $R, \bar{R} \in M_{I^2}$.



a) Write down the partition function of D.

b) Now assume that $R = G \otimes H$ and $\bar{R} = G^{-1} \otimes H^{-1}$ for invertible matrices $G, H \in M_I$. Show that in this case,

$$Z(D) = \mathrm{Tr}(G^2) \cdot \mathrm{Tr}(H^{-2}) \cdot \mathrm{Tr}(HG^{-1}HG^{-1})\,.$$

What do you get when $H = G = 1 \in M_I$?

Problem 15: Adjoints and tensor products

Let I_1, I_2, J_1, J_2 be finite index sets, and $U \in M_{I_1,J_1}$, $V \in M_{I_2,J_2}$ be arbitrary matrices. Using the graphical notation for matrices, verify that

$$(U \otimes V)^* = U^* \otimes V^*\,.$$

Problem 16: Temperley-Lieb algebra

Let I be a finite set of $n = |I|$ elements, and $E_{a,b} \in M_I$ the canonical matrix units (that is, $(E_{a,b})^i_j = \delta^i_a \delta^b_j$ for all $a, b, i, j \in I$). Define

$$U_1 := n^{-1/2} \sum_{i,j \in I} E_{ij} \otimes 1\,, \qquad U_2 := n^{1/2} \sum_{i \in I} E_{ii} \otimes E_{ii}\,,$$

and show that the following equations hold

$$U_1^2 = n^{1/2}\, U_1\,, \qquad U_2^2 = n^{1/2} U_2\,, \qquad U_2 U_1 U_2 = U_2\,, \qquad U_1 U_2 U_1 = U_1\,.$$

Problem 17 $\star$: Yang-Baxter equation

Let I be a finite index set and $R \in M_{I^2}$ a solution of the Yang-Baxter equation, i.e.

$$(R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R)\,.$$

Show that for invertible $A \in M_I$, also $S := (A \otimes A)R(A^{-1} \otimes A^{-1})$ is a solution of the Yang-Baxter equation. Show furthermore that if $S = R$, then also $T = (1 \otimes A)R(1 \otimes A^{-1})$ is a solution of the Yang-Baxter equation.