Thermal Equilibrium States for QFTs on Moyal Minkowski Spacetime

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joint work with Albert Huber and Jan Schlemmer

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ICMP 2012

– 07.08.2012 –

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    ω: translationally invariant state on $\mathcal{A}$, GNS rep π_ω, U_ω

$$\pi_\omega(A) \mapsto \pi_\omega(A)_\theta = (2\pi)^{-4} \int dp\, dx\, e^{ipx}\, U_\omega(\theta p)\pi_\omega(A) U_\omega(\theta p)^{-1}\, U_\omega(x)$$

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» For Lorentz-covariant theories, need full Lorentz orbit $\Theta = \{\Lambda\theta_0\Lambda^{-1} \,:\, \Lambda \in \mathcal{L}\}$ of θ's:

$$U_\omega(x,\Lambda)\pi_\omega(A)_\theta U_\omega(x,\Lambda)^{-1} = (U_\omega(x,\Lambda)\pi_\omega(A)U_\omega(x,\Lambda)^{-1})_{\Lambda\theta\Lambda^{-1}}$$

› Observable algebra $\mathcal{A}_\Theta$ generated by all $\pi_\omega(A)_\theta$, $A \in \mathcal{A}$, $\theta \in \Theta$

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- This localization makes it possible to do scattering theory
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Questions:

- What are the thermal equilibrium states (KMS) of a field theory deformed in the vacuum representation, if any?
  Dynamics: time translation $U_\omega(t)$. Inverse temperature $\beta > 0$ fixed.
- Deformation procedures producing wedge-local field theories without using spectrum condition?
- Thermal properties of QFTs on noncommutative spaces?

KMS states for Rieffel-deformed algebras

- Consider QFT in vacuum state ω (notation $A = \pi_\omega(A)$, $U = U_\omega$)
- $\mathcal{A}_\Theta \supset \mathcal{A}_{\{\theta\}} := \{A_\theta \,:\, A \in \mathcal{A}\} \cong (\mathcal{A}, \times_\theta)$ with Rieffel product $\times_\theta$ [Rieffel 92]

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Theorem

The translationally invariant KMS states at fixed temperature on $\mathcal{A}$ and $\mathcal{A}_{\{\theta\}}$ are in $1:1$ correspondence via $\omega_\beta^\theta(A_\theta) := \omega_\beta(A)$.

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» Study the situation in a concrete model: free massive scalar field in vacuum rep.

$$\mathcal{A} = \text{polynomials}(\phi(f)),\ f \in \mathscr{S}(\mathbb{R}^4)$$

$$[\phi(f), \phi(g)] = \int dp\, dq\, \tilde{f}(-p)\tilde{g}(-q)\, \tilde{C}(p,q) \cdot 1$$

- Formally:

$$\tilde{\phi}_\theta(p)\tilde{\phi}_{\theta'}(p') - e^{ip\cdot(\theta+\theta')p'}\tilde{\phi}_{\theta'}(p')\tilde{\phi}_\theta(p) = \tilde{C}(p,p')\boldsymbol{U((\theta' - \theta)p)}$$

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Form of KMS functionals

The set of all KMS functionals ω_β^Θ on $\mathcal{A}_\Theta$ is precisely the family of functionals given by the n-point distributions

$$\omega_\beta^\Theta(\tilde{\phi}_{\theta_1}(p_1)\cdots\tilde{\phi}_{\theta_{2n+1}}(p_{2n+1})) = 0\,,$$

$$\omega_\beta^\Theta(\tilde{\phi}_{\theta_1}(p_1)\cdots\tilde{\phi}_{\theta_{2n}}(p_{2n})) =$$

$$w(-\sum_{j=1}^{2n}\theta_j p_j)\prod_{1\leq l<r\leq n} e^{ip_l\cdot\theta_l p_r}\sum_{(\boldsymbol{l},\boldsymbol{r})\in\mathcal{C}_{2n}}\prod_{k=1}^{n}\frac{\tilde{C}(p_{l_k},p_{r_k})}{1-e^{\beta p_{l_k}^0+ip_{l_k}\cdot\sum_{b=1}^{2n}\theta_b p_b}}$$

with a largely undetermined function "$w(x)=\omega_\beta^\Theta(U(x))$"

- The ω_β^Θ are translationally and rotationally invariant, normalized: $\omega_\beta^\Theta(1)=1$, real: $\omega_\beta^\Theta(A^*)=\overline{\omega_\beta^\Theta(A)}$
- Calculation done using "twisted" KMS functionals [Buchholz/Longo 99]

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- Hard to explicitly see positivity on $\mathcal{A}_\Theta$. Numerics?
- If w continuous: Positivity of two-point function,

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Positivity by continuity of deformation

Let $A = \sum \phi(f_1) \cdots \phi(f_n) \in \mathcal{A}$ with $\omega_\beta(A^*A) > 0$ and $\hat{A} = \sum \phi(f_1)_{\theta_1} \cdots \phi(f_n)_{\theta_n}$. Then there exists $\varepsilon > 0$ such that

$$\omega_\beta^\Theta(\hat{A}^*\hat{A}) > 0$$

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Problem: ε depends on A.

Approximation by Gibbs states

Produce an explicit w-function by finite volume approximation

- Formulate deformed field theory in a box $[0, L]^3$ with periodic boundary conditions
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» thermodynamic limit:

$$w(x) = \lim_{L\to\infty} \omega_{\beta,L}(U_L(x)) = \begin{cases} 1 & ; \quad x = 0 \\ 0 & ; \quad x \neq 0 \end{cases}$$

$$\omega_\beta^\Theta(\tilde{\phi}_{\theta_1}(p_1) \cdots \tilde{\phi}_{\theta_{2n}}(p_{2n})) = \prod_{1\le l<r\le n} e^{ip_l\cdot\theta_l p_r} \sum_{(\boldsymbol{l},\boldsymbol{r})\in\mathcal{C}_{2n}^{\boldsymbol{\theta}}} \prod_{k=1}^{n} \frac{\tilde{C}(p_{l_k}, p_{r_k})}{1 - e^{\beta p_{l_k}^0}}$$

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For distinct $\theta_1, ..., \theta_n$:

$$\omega_\beta^\Theta((\phi_{\theta_1}(f_1)\cdots\phi_{\theta_n}(f_n))^*(\phi_{\theta_1}(f_1)\cdots\phi_{\theta_n}(f_n))) = \prod_{k=1}^{n} \omega_\beta(\phi(f_k)^*\phi(f_k))$$

Conclusion

- Covariant QFTs on Moyal Minkowski space with observable algebra $\mathcal{A}_\Theta = \bigvee_{\theta\in\Theta} \mathcal{A}_{\{\theta\}}$
- KMS states on each $\mathcal{A}_{\{\theta\}}$ are easy to get (positivity of Rieffel's deformation)
- KMS functionals on full $\mathcal{A}_\Theta$ can be computed.
- Positivity?
- Structure of set of KMS states on $\mathcal{A}_\Theta$ needs further investigation