Oscillatory Integrals for Vector-Valued Symbols and Locally Noncommutative Spacetimes

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joint work with Stefan Waldmann

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Motivation: Locally noncommutative spacetimes

Physics motivation:

- Combining classical gravity and quantum theory suggests that spacetime effectively becomes noncommutative at very small scales.
- However, most models of noncommutative spacetimes are noncommutative at all scales. (Example: Moyal space)
- Need models with "localized noncommutativity".

Mathematics motivation:

- Generalize Rieffel's strict deformation by actions of $\mathbb{R}^n$ to a much broader setting.

Idea for a "locally noncommutative" star product (say, on $\mathscr{S}(\mathbb{R}^n)$):
Use Rieffel's formula [Rieffel 1992]

$$(f \times_\theta g)(y) := (2\pi)^{-n} \int dp\, dx\, e^{i\langle p,x\rangle}\, f(\tau_{\theta p}(y)) g(\tau_x(y))$$

with an action $\tau : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ which acts non-trivially only inside a compact set K.

- in formal power series setting: [Bahns, Waldmann 2006]
- for continuous functions: [Heller, Neumaier, Waldmann 2006]

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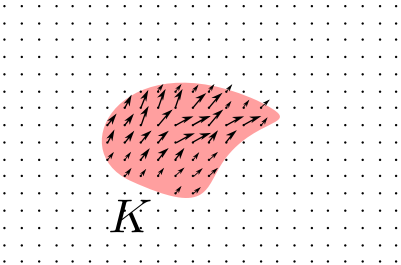

action τ_x

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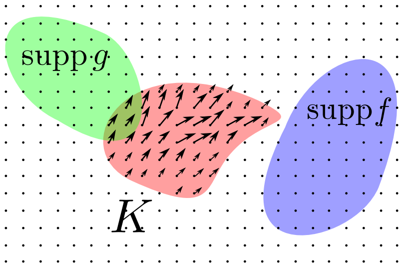
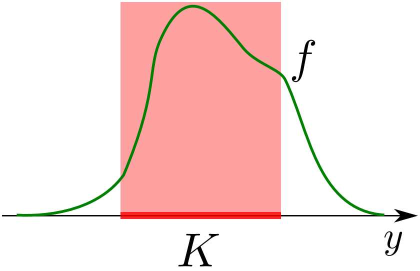


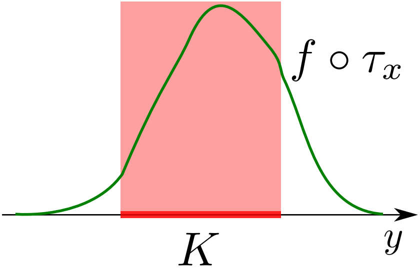
$f \circ \tau_x = f$ for all $x \in \mathbb{R}^d$

$\Rightarrow f \times_\theta g = f \cdot g$



Problem: Such actions $f \mapsto f \circ \tau_x$ are only isometric for $\sup$-norm, must be unbounded in x when considering derivatives.

- $\to$ Expand Rieffel's deformation procedure to cover this situation.



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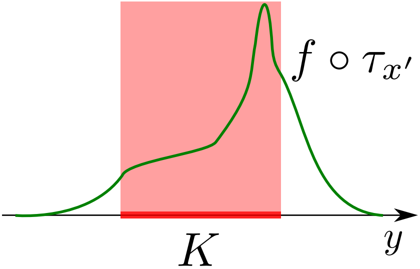

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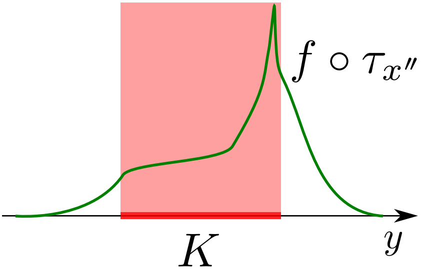

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Outline of the talk

1. Vector-valued symbols
2. Oscillatory integrals for vector-valued symbols
3. Rieffel deformations for smooth polynomially bounded $\mathbb{R}^n$-actions
4. Examples. Locally noncommutative spacetimes.
5. (Towards QFT on locally noncommutative spacetimes.)

Vector-Valued Symbols

- Target space V: locally convex sequentially complete vector space with defining system of seminorms $\mathcal{Q}$
- Symbol seminorms: $F \in C^\infty(\mathbb{R}^n, V)$, $m, \rho \in \mathbb{R}$, $\mathrm{q} \in \mathcal{Q}$, $\mu \in \mathbb{N}_0^n$,

$$\|F\|_{\mathrm{q},\mu}^{m,\rho} := \sup_{x \in \mathbb{R}^n} \frac{\mathrm{q}(\partial_x^\mu F(x))}{(1 + \|x\|^2)^{\frac{1}{2}(m - \rho|\mu|)}}$$

- Order and type for $\mathcal{Q}$: Arbitrary maps $\boldsymbol{m}, \boldsymbol{\rho} : \mathcal{Q} \to \mathbb{R}$

Definition

A symbol of order $\boldsymbol{m}$ and type $\boldsymbol{\rho}$ is a smooth function $F : \mathbb{R}^n \to V$ s.t.

$$\|F\|_{\mathrm{q},\mu}^{\boldsymbol{m}(\mathrm{q}),\boldsymbol{\rho}(\mathrm{q})} < \infty \quad \text{for all } \mathrm{q}, \mu.$$

Symbol spaces $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, V) \equiv \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(V) \equiv \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}$.

Basic properties of the symbol spaces

Let $\boldsymbol{m} \leq \boldsymbol{m}'$ orders and $\boldsymbol{\rho} \geq \boldsymbol{\rho}'$ types.

- $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}$ is a (sequentially) complete locally convex vector space.
- Continuous inclusions

$$C_0^\infty \subset \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}} \subset \mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}'} \subset C^\infty .$$

- Let $\boldsymbol{m}' > \boldsymbol{m}$ and $\boldsymbol{\rho}' \leq \min\{1, \boldsymbol{\rho}\}$. Then $C_0^\infty \subset \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}$ is sequentially dense in the $\mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}'}$-topology.

Some operations with symbols

- Partial derivatives $\partial_x^\mu : \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}} \to \mathrm{S}^{\boldsymbol{m}-\boldsymbol{\rho}|\mu|,\boldsymbol{\rho}}$ are linear + continuous.

- A contin. linear map $A : V \to W$ induces a contin. linear map

$$\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(V) \to \mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}'}(W)\,, \qquad F \mapsto (x \mapsto AF(x))$$

  for appropriate order and type $\boldsymbol{m}', \boldsymbol{\rho}'$ on W.

- A contin. bilinear map $\mu : V \times W \to U$ gives contin. bilinear map

$$\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(V) \times \mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}'}(W) \to \mathrm{S}^{\boldsymbol{m}'',\boldsymbol{\rho}''}(U)$$
$$(F, G) \mapsto (x \mapsto \mu(F(x), G(x)))$$

  for appropriate $\boldsymbol{m}'', \boldsymbol{\rho}''$ on U.

- ...

Oscillatory Integrals for Vector-Valued Symbols

Now take symbols $F \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^{2n}, V)$ and consider the integral

$$(2\pi)^{-n} \int_{\mathbb{R}^{2n}} dp\, dx\, e^{i\langle p,x\rangle} F(p,x)$$

for some non-degenerate bilinear form on $\mathbb{R}^n$ with determinant ± 1.

- How to define such an integral? F is smooth but does not decay.
- Degree $\boldsymbol{m}(\mathrm{q})$ of divergence of F may depend on seminorm q, also in an unbounded manner over $\mathcal{Q}$.

- Define first

$$I_0 : C_0^\infty \to V$$

as the corresponding Riemann integral. Works because F has compact support.

Lemma

Let $\boldsymbol{m}$ be an order and $-1 < \boldsymbol{\rho} \leq 1$ a type. Then $I_0 : C_0^\infty \to V$ is continuous if C_0^∞ is given the $\mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}$-topology.

(proof by integration by parts against $e^{i\langle p,x\rangle}$ and estimates)

Definition

Let $\boldsymbol{m} < \boldsymbol{m}'$ be orders and $-1 < \boldsymbol{\rho}' \leq \boldsymbol{\rho} \leq 1$ types. Then we define an oscillatory integral

$$I_{\boldsymbol{m}',\boldsymbol{\rho}'}^{\boldsymbol{m},\boldsymbol{\rho}} : \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}} \to V$$

by continuous extension from C_0^∞ in the $\mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}'}$-topology.

(Recall that $C_0^\infty \subset \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}$ is sequentially dense in the $\mathrm{S}^{\boldsymbol{m}',\boldsymbol{\rho}'}$-topology)

Let $\mathrm{S} := \mathrm{S}(\mathbb{R}^{2n}, V) := \bigcup_{\boldsymbol{m}, -1<\boldsymbol{\rho}\leq 1} \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^{2n}, V)$ and define

$$I : \mathrm{S} \to V$$
$$I(F) := I^{\boldsymbol{m},\boldsymbol{\rho}}_{\boldsymbol{m}',\boldsymbol{\rho}'}(F) \quad \text{for} \;\; F \in \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}},\; \boldsymbol{m}' > \boldsymbol{m},\; -1 < \boldsymbol{\rho}' \leq \boldsymbol{\rho} \leq 1.$$

Theorem

I is well-defined and independent of Q.

In practice, the oscillatory integral can be calculated as follows:

Lemma

Let $F \in \mathrm{S}(\mathbb{R}^{2n}, V)$ *and* $\chi \in C_0^\infty(\mathbb{R}^{2n}, \mathbb{R})$ *with* $\chi(p, x) = 1$ *in some open neighbourhood of* $(0, 0)$*. Then*

$$I(F) = (2\pi)^{-n} \lim_{\varepsilon \to 0} \int dp\, dx\, e^{i\langle p, x\rangle} \chi(\varepsilon p, \varepsilon x)\, F(p, x)\,.$$

Calculational rules for the oscillatory integral:

- continuous linear maps can be pulled under the integral
- integration by parts works as for Riemann integrals
- affine substitutions work as for Riemann integrals
- normalization: If F is constant, then

$$I(F) = F(0)\,.$$

- A Fubini theorem holds.

Rieffel deformations for smooth polynomially bounded $\mathbb{R}^n$-actions

Now application of oscillatory integrals to Rieffel-type deformations.

Definition

A smooth polynomially bounded $\mathbb{R}^n$-action (of order $\boldsymbol{m}$ and type 0) is an action $\alpha : \mathbb{R}^n \times V \to V$ such that

- $\alpha(v) : x \mapsto \alpha_x(v) \in \mathrm{S}^{\boldsymbol{m},0}$ for all $v \in V$.
- $V \ni v \mapsto \alpha(v) \in \mathrm{S}^{\boldsymbol{m},0}$ is continuous.

This implies that

- Each $\alpha_x : V \to V$ is continuous.
- $V = V^\infty$ as locally convex spaces.

Examples: $V := \mathrm{S}^{\boldsymbol{m},\boldsymbol{\rho}}(\mathbb{R}^n, W)$, $\boldsymbol{\rho} \geq 0$ with $(\alpha_x F)(y) := F(x+y)$.

Now consider three target spaces V, W, U, with smooth pol. bnd. actions α^V, α^W, α^U, and a (separately) continuous bilinear map

$$\mu : V \times W \to U$$

such that

$$\alpha_x^U \mu(v, w) = \mu(\alpha_x^V v, \alpha_x^W w) \,.$$

Examples:

- $V = W = U$ algebra with product μ and automorphic action $\alpha^V = \alpha^W = \alpha^U$.
- V algebra and $W = U$ left V-module with module map $\mu : V \times W \to W$ and compatible actions. (covariant rep.)

Aim: Deform μ with oscillatory integral formula.

For $v \in V$, $w \in W$, and a deformation parameter $\theta \in \mathbb{R}^{n\times n}$, let

$$\mu^\theta_{vw} : \mathbb{R}^{2n} \to U\,, \qquad \mu^\theta_{vw}(p,x) := \mu(\alpha^V_{\theta p} v, \alpha^W_x w)\,.$$

Proposition:

- $\mu^\theta_{vw} \in \mathrm{S}(\mathbb{R}^{2n}, U)$.
- The maps

$$V \times W \ni (v,w) \longmapsto I(\mu^\theta_{vw}) =: \mu_\theta(v,w)$$

  are bilinear and (separately) continuous.
- μ_θ is still compatible with the actions α^V, α^W, α^U.
- $\mu_0 = \mu$.
- $(\mu_\theta)_{\theta'} = \mu_{\theta+\theta'}$.

Deformations of algebras

- $A := V = W = U$ locally convex sequentially complete algebra with (separately) continuous product μ
- $\alpha := \alpha^V = \alpha^W = \alpha^U$ automorphic $\mathbb{R}^n$-action, smooth and polynomially bounded.

Theorem

- *If μ is associative, also μ_θ is associative.*
- *If μ has an identity 1, this is also an identity for μ_θ, i.e. $\mu_\theta(a, 1) = a = \mu_\theta(1, a)$.*
- *If A is a *-algebra and θ is skew-symmetric, then*

$$\mu_\theta(a, b)^* = \mu_\theta(b^*, a^*)\,.$$

For $V = \mathscr{S}(\mathbb{R}^n)$ with $\mu =$ pointwise product and $\alpha =$translations: [Gayral, Gracia-Bondia, Iochum, Schücker, Varilly 2004]

Deformations of covariant modules

- $A := V$ loc. conv. sequ. compl. algebra with (separately) continuous product μ, and smooth polynomially bounded $\mathbb{R}^n$-action $\alpha := \alpha^V$
- $E := W = U$ loc. conv. sequ. compl. left A-module with (separately) continuous module map $\tilde{\mu}$, and smooth polynomially bounded $\mathbb{R}^n$-action $\beta := \alpha^W = \alpha^U$ such that

$$\beta_x \tilde{\mu}(a, \psi) = \tilde{\mu}(\alpha_x a, \beta_x \psi)$$

Theorem

- $(E, \tilde{\mu}_\theta)$ *is a left* (A, μ_θ)*-module, i.e.*

$$\tilde{\mu}_\theta(a, \tilde{\mu}_\theta(b, \psi)) = \tilde{\mu}_\theta(\mu_\theta(a, b), \psi)\,.$$

  (In C^*-setting: [Buchholz, GL, Summers 2010])
- *If ψ is β-invariant, then $\tilde{\mu}_\theta(a, \psi) = \tilde{\mu}(a, \psi)$.*
- *In Hilbert space situation: $\tilde{\mu}_\theta(a, \,\cdot\,)^* \supset \tilde{\mu}_\theta(a^*, \,\cdot\,)$.*

Compactly supported $\mathbb{R}^n$-actions

Now back to our initial example of compactly support $\mathbb{R}^n$-actions.

Definition

A smooth $\mathbb{R}^n$-action with support in $K \subset \mathbb{R}^n$ is a smooth function $\tau : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that

- $\tau_x(\tau_{x'}(y)) = \tau_{x+x'}(y)$ for all $x, x', y \in \mathbb{R}^n$.
- $\tau_x(y) = y$ for all $x \in \mathbb{R}^n$ if $y \notin K$.
- There exists a sequence $\{b_l\}_{l\in\mathbb{N}_0} \subset \mathbb{R}_+$, and for each $\mu \in \mathbb{N}_0^n$, $l \in \mathbb{N}_0$, there exists $c_{l\mu} > 0$, such that

  $$\sup_{y\in K, |\nu|\leq l} \|\partial_y^\nu \partial_x^\mu \tau_x(y)\| \leq c_{l\mu}(1 + \|x\|^2)^{\frac{1}{2}b_l}$$

  for all $x \in \mathbb{R}^n$.

Proposition

Let τ be a smooth compactly supported $\mathbb{R}^n$-action. Then

$$\alpha : \mathbb{R}^n \times C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)\,, \qquad \alpha_x(f) := f \circ \tau_x$$

is a smooth polynomially bounded $\mathbb{R}^n$-action on $C^\infty(\mathbb{R}^n)$.

Works also for other function spaces such as $\mathrm{S}^{m,\rho}$ or $\mathscr{S}$.

How to construct τ

Idea: Consider $n = 1$ first, and

$$\tau_x(y) := \begin{cases} \gamma^{-1}(\gamma(y) + x) & ; \quad |y| < 1 \\ y & ; \quad |y| \geq 1 \end{cases}, \qquad x \in \mathbb{R},$$

with diffeomorphism $\gamma : (-1, 1) \to \mathbb{R}$. [Weinstein]

- τ action (clear)
- τ smooth (easy to guarantee, use ODE theorems)
- bounds on derivatives of τ: tricky. Have to build suitable γ.

Theorem

Let $K \subset \mathbb{R}^n$ be open. Then there exist non-trivial smooth $\mathbb{R}^n$-actions with support in K.

Locally noncommutative spaces

- Consider for example $C^\infty(\mathbb{R}^n)$ with action $\alpha_x f := f \circ \tau_x$ for some smooth compactly supported τ.
- Consider Rieffel deformation $C^\infty(\mathbb{R}^n)^\alpha_\theta$,

$$f \times^\alpha_\theta g = (2\pi)^{-n} \int dp\, dx\, e^{i\langle p,x\rangle} (\alpha_{\theta p} f) \cdot (\alpha_x g)$$

Then

$$f \times^\alpha_\theta g = f \cdot g \quad \text{if} \quad \operatorname{supp} f \cap K = \emptyset \quad \text{or} \quad \operatorname{supp} g \cap K = \emptyset$$

"$\mathbb{R}^n$ becomes noncommutative only in the region K"

Outlook: Towards QFT models on locally noncommutative spacetimes

- Consider QFT in its Wightman formulation, based on vacuum representations of the tensor algebra $\underline{\mathscr{S}}$ over $\mathscr{S}(\mathbb{R}^n)$ (Borchers-Uhlmann algebra).
- Model "noncommutative region" K by a smooth pol. bnd. action α^K on $\underline{\mathscr{S}}$, derived from a compactly supported action τ^K on $\mathbb{R}^n$. (for Moyal space: [Grosse, GL 2008])

$$(\alpha_x^K \underline{f})_n(y_1, ..., y_n) = \underline{f}_n(\tau_x^K(y_1), ..., \tau_x^K(y_n))$$

- Consider deformation of tensor product

$$\mu(\underline{f}, \underline{g}) = \underline{f} \otimes \underline{g}$$

on $\underline{\mathscr{S}}$ (only LF-algebra, but covered by our formalism)

Outlook: Towards QFT models on locally noncommutative spacetimes

- States ω on $\underline{\mathscr{S}}$ are not states for deformed product μ_θ (not positive)
- Have to deform $\omega \to \hat{\omega}$ to formulate states on $(\underline{\mathscr{S}}, \mu_\theta)$, using techniques from [Kaschek, Neumaier, Waldmann 2009]
- $\to$ deformed quantum fields $\phi^\theta_{\hat{\omega}}(\underline{f})$ in GNS-rep of $((\underline{\mathscr{S}}, \mu_\theta), \hat{\omega})$
- Study effects of the localized noncommutativity: Locally acausal / non-covariant behaviour.
- For $\underline{f}, \underline{g} \in \underline{\mathscr{S}}$ with $\operatorname{supp} \underline{f}$ spacelike to $\operatorname{supp} \underline{g}$:

$$[\phi^\theta_{\hat{\omega}}(\underline{f}),\, \phi^\theta_{\hat{\omega}}(\underline{g})] = 0 \quad \text{if } \operatorname{supp} \underline{f} \cap K = \emptyset \text{ or } \operatorname{supp} \underline{g} \cap K = \emptyset$$

- Particle scattering at the noncommutativity?
  [Work in progress with Stefan Waldmann], cf [Borris, Verch 2010]