Multiplicative deformations of the Borchers-Uhlmann algebra

arXiv 1104.1948
to appear in CMP

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1 July, 2011 - Göttingen
LQP 28

Motivation

Main focus of algebraic QFT since its invention:

- Model-independent analysis of QFT (nets of operator algebras)
- Development of tools for computing physically interesting quantities once a model is given

More recently: Construction of models with tools of algebraic QFT

This talk: Construction of models by algebraic deformation methods

Construction of QFT models via wedge algebras

Wedge algebras:

Baumgärtel/Wollenberg 92 Buchholz/Summers 08

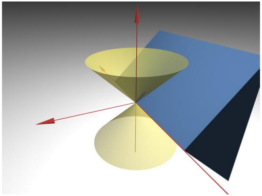
Input: Hilbert space $\mathcal{H}$ with representation U of the Poincaré group and vacuum vector Ω.

Wanted: $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ von Neumann "wedge algebra":

- $U(x,\Lambda)\mathcal{M}U(x,\Lambda)^{-1} \subset \mathcal{M}$ for $(x,\Lambda) \in \mathcal{P}_+$ with $\Lambda W_0 + x \subset W_0$
- $U(0,j)\mathcal{M}U(0,j)^{-1} \subset \mathcal{M}'$
- Ω cyclic for $\mathcal{M}$

with

- $W_0 = \{x \in \mathbb{R}^d \ : \ x_1 > |x_0|\} =$ right wedge
- reflection
  $j(x^0, .., x^{d-1}) = (-x^0, -x^1, x^2, .., x^{d-1})$



- With such data $U, \mathcal{M}$, a QFT model can be generated.

Deformations of wedge algebras

- One approach to finding such pairs $U, \mathcal{M}$ is to start with data $U_0, \mathcal{M}_0$ defining a free field theory, and search for solutions $U_0, \mathcal{M}$ by **deforming** $\mathcal{M}_0 \rightsquigarrow \mathcal{M}$ in a certain manner.
- A two-parameter family of solutions is known (warping) ▸ Grosse/GL 07 ▸ Buchholz/Summers 08 ▸ Buchholz/GL/Summers 10
- Applications of this method: ▸ Dappiaggi/GL/Morfa-Morales 10 ▸ Dybalski/Tanimoto 10 ▸ Morfa-Morales 11
- Other deformations ▸ Longo/Witten 10 ▸ Longo/Rehren 11 ▸ Tanimoto ??

Here:

Many more deformations, constructed in a Wightman framework.

The Borchers-Uhlmann algebra $\underline{\mathscr{S}}$

- $\underline{\mathscr{S}} = \bigoplus_{n=0}^{\infty} \mathscr{S}_n$ with $\mathscr{S}_n := \mathscr{S}(\mathbb{R}^{nd})$, $\mathscr{S}_0 := \mathbb{C}$, as top. space.
- elements $f = (f_0, f_1, ...f_N, 0...)$, $f_n \in \mathscr{S}_n$
- product

$$(f \otimes g)_n(x_1, ..., x_n) = \sum_{k=0}^{n} f_k(x_1, ..., x_k)\, g_{n-k}(x_{k+1}, ..., x_n)\,.$$

- identity $1_n := \delta_{n,0}$
- *-involution

$$f^*{}_n(x_1, ..., x_n) = \overline{f_n(x_n, ..., x_1)}\,.$$

- automorphic Poincaré action

$$(\alpha_{a,\Lambda} f)_n(x_1, ..., x_n) = {}^{(}\overline{f_n(\Lambda^{-1}(x_1 - a), ..., \Lambda^{-1}(x_n - a))}^{)}$$

Products on $\mathscr{S}$

- Deformations of algebras Gerstenhaber 64: deform the product
- **product** := bilinear associative separately continuous map $\mathscr{S} \times \mathscr{S} \to \mathscr{S}$, $f, g \mapsto f \hat{\otimes} g$ with

$$(f \hat{\otimes} g)^* = g^* \hat{\otimes} f^*, \qquad f \hat{\otimes} 1 = f = 1 \hat{\otimes} f\,.$$

- As a tensor algebra ($\cong$ free algebra), $\mathscr{S}$ is **rigid**. Its products are

$$f \hat{\otimes} g = \rho^{-1}(\rho(f) \otimes \rho(g))$$

  Only trivial deformations of $\mathscr{S}$ as an associative algebra exist.
- **But:** Situation very different for **nets** in $\mathscr{S}$. Here also products $f \otimes_\rho g := \rho^{-1}(\rho(f) \otimes \rho(g))$ give rise to non-trivial deformations.

Deformation maps and compatible states

Definition:

A deformation map (relative to W_0) is a linear homeomorphism $\rho : \mathscr{S} \to \mathscr{S}$ such that

- $\rho(1) = 1$.
- $\rho(f)^* = \rho(f^*)$.
- $\rho \circ \alpha_{x,\Lambda} = \alpha_{x,\Lambda} \circ \rho$ for all $(x, \Lambda) \in \mathcal{P}_+$ with $\Lambda W_0 + x \subset W_0$.

- Deformation maps form a group $\mathcal{R}$.
- ρ is **not** invariant under **all** Lorentz transformations,

$$\rho_\Lambda := \alpha_\Lambda \circ \rho \circ \alpha_\Lambda^{-1} \neq \rho \,.$$

- Need to represent the different product structures $\otimes_{\rho_\Lambda}$ (corresponding to different wedges) on the same Hilbert space.

Deformation maps and compatible states

Definition:

A state ω on $\underline{\mathscr{L}}$ is called **compatible** with a deformation map ρ if

$$\omega(f \otimes_\rho g) = \omega(f \otimes g)\,, \qquad f, g \in \underline{\mathscr{L}}\,.$$

Lemma:

Let ρ be a deformation map and ω a ρ-compatible state. Then ω is also a state on $(\underline{\mathscr{L}}, \otimes_\rho)$, and the GNS triples $(\mathcal{H}_\omega, \phi_\omega, \Omega_\omega)$ of $(\underline{\mathscr{L}}, \otimes)$ and $(\mathcal{H}^\rho_\omega, \phi^\rho_\omega, \Omega^\rho_\omega)$ of $(\underline{\mathscr{L}}, \otimes_\rho)$ are related by

$$\mathcal{H}^\rho_\omega = \mathcal{H}_\omega\,,$$
$$\Omega^\rho_\omega = \Omega_\omega\,,$$
$$\phi^\rho_\omega(f)\phi_\omega(g)\Omega_\omega = \phi_\omega(f \otimes_\rho g)\Omega_\omega\,, \qquad f, g \in \underline{\mathscr{L}}\,.$$

Deformation maps and compatible states

A ρ-compatible Poincaré invariant state ω defines deformed fields

$$\phi_\omega^{\rho_\Lambda}(f)\,, \qquad f \in \underline{\mathscr{S}}, \quad \rho_\Lambda := \alpha_\Lambda \circ \rho \circ \alpha_\Lambda^{-1},$$

as operators on the GNS space of $(\underline{\mathscr{S}}, \otimes), \omega$, with

$$\begin{aligned} \phi_\omega^{\rho_\Lambda}(f)\phi_\omega^{\rho_\Lambda}(g) &= \phi_\omega^{\rho_\Lambda}(f \otimes_{\rho_\Lambda} g) \\ \phi_\omega^{\rho_\Lambda}(f)^* &\supset \phi_\omega^{\rho_\Lambda}(f^*) \\ U_\omega(x,\Lambda)\phi_\omega^{\rho}(f)U_\omega(x,\Lambda)^{-1} &= \phi_\omega^{\rho_\Lambda}(\alpha_{x,\Lambda} f) \\ \phi_\omega^{\rho_\Lambda}(f)\Omega_\omega &= \phi_\omega(f)\Omega_\omega \end{aligned}$$

The *-algebra $\mathscr{P}_{\omega,0}^{\rho}$ generated by all $\phi_\omega^{\rho}(f)$, supp $f_n \subset W_0^n$ satisfies

$$U_\omega(x,\Lambda)\mathscr{P}_{\omega,0}^{\rho}U_\omega(x,\Lambda)^{-1} \subset \mathscr{P}_{\omega,0}^{\rho} \;\text{ for } \Lambda W_0 + x \subset W_0$$

and Ω_ω is cyclic for $\mathscr{P}_{\omega,0}^{\rho}$ if it is cyclic for $\mathscr{P}_{\omega,0}^{\mathrm{id}}$.

Procedure for deforming wedge-local QFT

- Find (class of) deformation maps ρ on $\underline{\mathscr{S}}$
- Find states compatible with ρ (at least free field Wightman state should be compatible)
- Adjust ρ and ω in such a way that ρ **is wedge-local in** ω:

$$[\phi_\omega^\rho(f),\, \phi_\omega^{\rho j}(g)]\Psi = 0,$$

$f \in \underline{\mathscr{S}}(W_0)$, $g \in \underline{\mathscr{S}}(jW_0)$, $\Psi \in \phi_\omega(\underline{\mathscr{S}})\Omega_\omega$.

- Possibly proceed to von Neumann algebras of bounded functions of field operators, $\mathcal{M}_\omega^\rho := \{\exp i\phi_\omega^\rho(f) \,:\, f \in \underline{\mathscr{S}}_{\mathbb{R}}(W_0)\}''$.
- Then $\mathcal{M}_\omega^\rho$, U_ω satisfy our requirements, and generate a QFT.

Rest of the talk: Carry out this program for a simple class of ρ's.

Multiplicative deformations

- A deformation map is called **multiplicative** (in momentum space) if it has the form

$$\widetilde{\rho(f)}_n(p_1, \dots, p_n) = \rho_n(p_1, \dots, p_n) \cdot \tilde{f}_n(p_1, \dots, p_n) \,.$$

$\rightarrow$ abelian subgroup of deformations $\mathcal{R}_0 \subset \mathcal{R}$.

Lemma:

For ρ to be a deformation map, the ρ_n must be smooth and satisfy

- polynomial bounds from above and below on their derivatives
- For each Lorentz transformation Λ with $\Lambda W_0 = W_0$,

$$\rho_n(\Lambda p_1, \dots, \Lambda p_n) = \rho_n(p_1, \dots, p_n) \,.$$

- $\overline{\rho_n(-p_n, \dots, -p_1)} = \rho_n(p_1, \dots, p_n)$ **(for $(f \otimes_\rho g)^* = g^* \otimes_\rho f^*$)**
- $\rho_0 = 1$ **(for $f \otimes_\rho 1 = f = 1 \otimes_\rho f$)**

Multiplicative deformations and compatible states

- The deformed product associated with $\rho \in \mathcal{R}_0$ is

$$(f \widetilde{\otimes_\rho} g)_n(p_1, .., p_n) = \sum_{k=0}^{n} \rho_{kn}(p_1, ..., p_n) \cdot \tilde{f}_k(p_1, .., p_k)\tilde{g}_{n-k}(p_{k+1}, .., p_n)$$

with $\rho_{kn}(p_1, ..., p_n) := \dfrac{\rho_k(p_1, .., p_k)\rho_{n-k}(p_{k+1}, .., p_n)}{\rho_n(p_1, .., p_n)}$

- Can take $\rho_1(p_1) = 1$ without loss of generality.
- Compatible states: Need

$$\int d\boldsymbol{p}\, \tilde{\omega}_n(-p_1, ..., -p_n)\, \rho_{kn}(p_1, .., p_n)\tilde{f}_k(p_1, .., p_k)\tilde{g}_{n-k}(p_{k+1}, .., p_n)$$

$$= \int d\boldsymbol{p}\, \tilde{\omega}_n(-p_1, ..., -p_n)\, \tilde{f}_k(p_1, .., p_k)\tilde{g}_{n-k}(p_{k+1}, .., p_n)$$

Multiplicative deformations and compatible states

- If $\rho \in \mathcal{R}_0$ is compatible with all **translationally invariant states**, then

$$\rho_n(p_1, \ldots, p_n) = \rho_k(p_1, \ldots, p_k)\rho_{n-k}(p_{k+1}, \ldots, p_n)$$

  for all $k \in \{1, \ldots, n-1\}$ and $p_1 + \ldots + p_n = 0$.
- One solution is known (warped convolution)
- More solutions? ... ▸ Alazzawi

Deformation two-point functions

Proposition:

Let $\rho_2 \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be a two-point function of a multiplicative deformation satisfying

$$\rho_2(p, -p) = 1\,, \qquad \rho_2(-p, q) = \rho_2(p, -q) = \rho_2(q, p) = \rho_2(p, q)^{-1}\,.$$

Define

$$\rho_0 := 1\,, \qquad \rho_1(p_1) := 1\,, \qquad \rho_n(p_1, \ldots, p_n) = \prod_{1 \leq l < r \leq n} \rho_2(p_l, p_r)\,, \quad n \geq 2.$$

Then

- The ρ_n define a multiplicative deformation $\rho \in \mathcal{R}_0$.
- ρ is compatible with all **quasi-free** translationally inv. states ω.

Problem of finding deformations reduced to two-point function ρ_2.

Definition:

A **deformation function** is a smooth function $R : \mathbb{R} \to \mathbb{C}$, polynomially bounded in all derivatives, such that

- $R(a)^{-1} = \overline{R(a)} = R(-a)\,, \qquad R(0) = 1.$
- The Fourier transform $\tilde{R}$ of R has support on the positive half line.

- Support restriction on $\tilde{R} \Leftrightarrow$ analyticity and boundedness of R.
- Examples:

$$R(a) = e^{ica} \prod_{k=1}^{N} \frac{z_k - a}{z_k + a}\,, \qquad c \geq 0,\ \mathrm{Im} z_1, \ldots, \mathrm{Im} z_N \geq 0$$

  These functions appear in many places ▸ Longo/Witten 10 ▸ Longo/Rehren 11
- Special case: $R(a) = e^{ia}$

  ▸ Grosse/GL 07 ▸ Buchholz/Summers 08 ▸ Grosse/GL 08 ▸ Buchholz/GL/Summers 10

Definition

A real $(d \times d)$-matrix Q is called **admissible** Buchholz/GL/Summers 10 if

- Q is antisymmetric w.r.t. the Minkowski inner product
- $\Lambda Q \Lambda^{-1} = \begin{cases} Q & ; \quad \Lambda \in \mathcal{L}_+^\uparrow \text{ with } \Lambda W_0 = W_0 \\ -Q & ; \quad \Lambda \in \mathcal{L}_+^\downarrow \text{ with } \Lambda W_0 = W_0 \end{cases}$
- $QV^+ \subset W_0$.

Q must have the form

$$Q = \begin{pmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa' \\ 0 & 0 & -\kappa' & 0 \end{pmatrix} \qquad \text{with } \kappa \geq 0, \ \kappa' \in \mathbb{R} \,.$$

Lemma:

Let R be a deformation function and Q an admissible matrix. Then the deformation two-point function

$$\rho_2(p, q) := R(-p \cdot Qq)$$

defines a multiplicative deformation $\rho(R, Q) \in \mathcal{R}_0$ compatible with all quasi-free translationally invariant states.

- Transformation law:

$$\alpha_{x,\Lambda}(f \otimes_{\rho(R,Q)} g) = \alpha_{x,\Lambda}(f) \otimes_{\rho(R,\pm\Lambda Q\Lambda^{-1})} \alpha_{x,\Lambda}(g)$$

Wedge-Locality

Theorem:

Let R be a deformation function, and Q an admissible matrix. Then the deformation $\rho(R, Q)$ is wedge-local in any quasi-free Wightman state.

Important ingredients of proof:

- Need to show that for $f \in \underline{\mathscr{S}}(W_0)$, $g' \in \underline{\mathscr{S}}(W_0')$, $h, k \in \underline{\mathscr{S}}$,

$$\omega((h \otimes_\rho f) \otimes (g' \otimes_{\rho'} k)) = \omega((h \otimes_{\rho'} g') \otimes (f \otimes_\rho k)).$$

- The automorphisms

$$(\widetilde{\tau_x^R f})_n(p_1, \ldots p_n) := \prod_{k=1}^{n} R(x \cdot p_k) \cdot \tilde{f}_n(p_1, \ldots, p_n)$$

satisfy

$$\tau_x^R(\underline{\mathscr{S}}(W_0)) \subset \underline{\mathscr{S}}(W_0) \ \text{ for } \ x \in \overline{W_0}\,,$$

$$\text{supp}\,(h_m \otimes_\rho f_n) \subset \mathbb{R}^{md} \times W_0^n \ \text{ for supp}\, \tilde{h}_m \subset V_-^m, \quad \text{supp}\, f_n \subset W_0^n\,.$$

Fock space representations

- Quasi-free Wightman state $\omega \to$ Fock space $\mathcal{H}_\omega$ as GNS space.
- If $\tilde{\omega}_2(p,q) = \delta(p+q)\delta(p^0 - \varepsilon_{\boldsymbol{p}})\varepsilon_{\boldsymbol{p}}^{-1}$, $\varepsilon_{\boldsymbol{p}} = (\boldsymbol{p}^2 + m^2)^{1/2}$, $m > 0$, then ϕ_ω is the scalar massive Klein-Gordon field.

Deformed field $\phi_{R,Q} := \phi_\omega^{\rho(R,Q)}$ satisfies by construction:

- $U(x,\Lambda)\phi_{R,Q}(f)U(x,\Lambda)^{-1} = \phi_{R,\Lambda Q \Lambda^{-1}}(\alpha_{x,\Lambda} f)$ for $(x,\Lambda) \in \mathcal{P}_+^\uparrow$.
- $U(0,j)\phi_{R,Q}(f)U(0,j)^{-1} = \phi_{R,-Q}(\alpha_j f)$.
- $[\phi_{R,Q}(f),\, \phi_{R,-Q}(g)]\Psi = 0$ for $f \in \underline{\mathscr{S}}(W_0)$, $g \in \underline{\mathscr{S}}(W_0')$, $\Psi \in \mathcal{D}$.
- still solves Klein-Gordon equation
- $\lambda \mapsto \phi_{R,\lambda \cdot Q}(f)\Psi$ continuous, $\Psi \in \mathcal{D}$. In particular

$$\lim_{\lambda \to 0} \phi_{R,\lambda\cdot Q}(f)\Psi = \phi(f)\Psi\,.$$

Von Neumann algebras and modular data

With $\mathscr{P}_R(\Lambda W_0 + x) :=$ *-alg. gen. by $\phi_{R,\Lambda Q \Lambda^{-1}}(f)$, $f \in \underline{\mathscr{S}}(\Lambda W_0 + x)$,

$$W \longmapsto \mathscr{P}_R(W)$$

is an isotonous, covariant, (wedge-) local net of *-algebras satisfying the Reeh-Schlieder property.

- Passage to von Neumann algebras $\mathcal{M}_R(W)$ possible here (essentially self-adjoint field operators).
- Ω is cyclic and separating for $\mathcal{M}_R(W_0)$, with modular data independent of R.
- Infinite family of solutions to inverse problem in modular theory
  ▸ Wollenberg
- Ω is not cyclic for spacelike cones.

Deformed field operators

- Explicit realization of deformed fields: Replace creation/annihilation operators $a^{\#}(p)$ by

$$a_{R,Q}(p) := a(p)\, T_R(Qp)\,, \qquad a^{\dagger}_{R,Q}(p) := a^{\dagger}(p)\, T_R(-Qp)\,,$$

$$(T_R(x)\Psi)_n(q_1, \ldots, q_n) := \prod_{k=1}^{n} R(x \cdot q_k) \cdot \Psi_n(q_1, \ldots, q_n)\,.$$

- For $f_1 \in \mathscr{S}_1$, the deformed fields take the form

$$\phi_{R,Q}(f_1) = (2\pi)^{-d} \int dp\, dx\, e^{-ip\cdot x}\, U(x,1)\phi(f_1)U(-x,1)\, T_R(-Qp)$$

  Formula similar to warped convolution, but not valid for higher tensors $f_n \in \mathscr{S}_n$, $n > 1$.

Two-particle scattering

- The S-matrix for $2 \to 2$ processes can be calculated. For $f_1, g_1, h_1, k_1 \in \mathscr{S}_1$ with velocity supports $\Gamma(f_1) - \Gamma(g_1) \subset W_0$, $\Gamma(h_1) - \Gamma(k_1) \subset W_0$,

$$\langle f_1^+ \times_{\text{out}}^R g_1^+ ,\, h_1^+ \times_{\text{in}}^R k_1^+ \rangle$$
$$= \int d\mu(p_1)\, d\mu(p_2)\, R(-p_1 \cdot Qp_2)^2\, \overline{\tilde{f}_1(p_1)}\,\overline{\tilde{g}_1(p_2)}\tilde{h}_1(p_1)\widetilde{k}_1(p_2)\,.$$

- Phase shift effect, no particle production
- S-matrix fully Lorentz invariant only in $d = 1 + 1$
- The deformation function is the root of the two-particle S-matrix

Deformations of QFTs and integrable models

- When deforming in $d = 1 + 1$, one obtains integrable models.
- rapidity-parametrized on-shell momenta $p(\theta) = m(\cosh\theta, \sinh\theta)$ lead to

$$S(\theta_1 - \theta_2) := R(-p(\theta_1) \cdot Qp(\theta_2))^2 = R(\kappa m^2 \sinh(\theta_1 - \theta_2))^2$$

  with the usual unitarity, analyticity, crossing relations of the scattering function S.
- Deformed ann./creation operators $\rightarrow$ **Zamolodchikov algebra**
- Same structure as in algebraic construction of integrable models
  ▸ Schroer 97 ▸ Schroer 00 ▸ GL 03 ▸ Buchholz/GL 04 ▸ GL 06

Theorem:

Every 2d integrable QFT with scattering function S of the above form arises from a free field theory by deformation. If S is **regular**, then the deformed theory satisfies Reeh-Schlieder for double cones above a minimal size ▸ L 06 .

Conclusion

- Multiplicative deformations: Large family of new QFT deformations
- Lead to wedge-local covariant models in any dimension
- In $d = 1 + 1$, provide deformations of free field theories into fully local integrable QFTs
- In $d > 1 + 1$, more complicated deformations (non-multiplicative) for more realistic interaction needed (particle production).
- Need deformation maps of the form

$$\widetilde{\rho(f)}_n(p_1, .., p_n) = \sum_{m=0}^{\infty} \int dq_1 .. dq_m \, \rho_{nm}(p_1, .., p_n; q_1, .., q_m) \, \tilde{f}_m(q_1, .., q_m)$$

- More complicated, but might give interacting models in $d > 1 + 1$ (wedge-local fields no longer polarization-free)