Star products on Hilbert space

Gandalf Lechner

joint work with D. Buchholz and S. Summers

Department of Physics, University of Vienna

Bayrischzell 2010

- Moyal-Weyl star product: $f, g \in S(\mathbb{R}^d)$, $\theta \in \mathbb{R}_-^{d\times d}$,

$$(f \times_\theta g)(x) = (2\pi)^{-d} \iint du\, dv\, e^{-iuv}\, f(x-\theta u) g(x-v)$$

- Rieffel version: C^*-algebra $\mathcal{A}$ with strongly continuous automorphic $\mathbb{R}^d$-action α.

$$A \times_\theta B = (2\pi)^{-d} \iint du\, dv\, e^{-iuv}\, \alpha_{\theta u}(A)\alpha_v(B)$$

- deformation changes product " $\cdot$ " $\rightarrow$ " $\times_\theta$ ",
  and keeps algebra elements $A \in \mathcal{A}$
- **This talk:** 1) Describe a deformation that keeps the product, and changes the algebra elements $A \rightarrow A_\theta$
- 2) apply to deformations of quantum field theories [with H. Grosse]

- General setting: $\mathcal{H}$ Hilbert space, U unitary rep. of $\mathbb{R}^d$ on $\mathcal{H}$.
- Deformation of operator $A \in \mathcal{B}(\mathcal{H})$:

$$A_\theta = (2\pi)^{-d} \iint dx\, dy\, e^{-ixy}\, U(\theta x) A U(-\theta x + y)$$

warped convolution of A

- earlier version: $A_\theta = \int dE(x) U(\theta x) A U(-\theta x)$ [Buchholz/Summers 08]
- integrals can be defined as bounded operators for smooth A.
- $\rightarrow$ deformation of smooth algebra $\mathcal{C}^\infty \subset \mathcal{B}(\mathcal{H})$

Properties of warped convolutions

How does the algebraic structure change?

- $A \mapsto A_\theta$ is linear
- $A_\theta B_\theta = (A \times_\theta B)_\theta$
- $(A_\theta)^* = (A^*)_\theta$
- $1_\theta = 1$

$A \mapsto A_\theta$ **extends to rep. of Rieffel-deformed C^*-alg. $(\mathcal{C}_\theta, \times_\theta)$.**

- Deformed operators A_θ with different θ are represented on the same space. Useful for discussing covariance and locality.
- Example: Let A, B such that
  $[U(\theta x)AU(-\theta x),\ U(-\theta y)BU(\theta y)] = 0$ for all $x, y \in \mathrm{spec}\, U$.
  Then
  $$[A_\theta,\ B_{-\theta}] = 0\,.$$

- First examples with Harald Grosse: deformed CCR:

$$a(p)_\theta = a(p)U(\theta p), \qquad a^*(p)_\theta = a^*(p)U(-\theta p)$$

describes scalar quantum field on NC Minkowski with θ-dependent scattering [Grosse, L 07]

- Deformations of general (Wightman) quantum fields $\phi(x)$, new n-point functions

$$\begin{aligned} W_n(x_1, ..., x_n) &= \langle \Omega,\, \phi(x_1) \cdots \phi(x_n)\Omega \rangle \\ \to W_n^\theta(x_1, ..., x_n) &= \langle \Omega,\, \phi(x_1)_\theta \cdots \phi(x_n)_\theta\Omega \rangle \end{aligned}$$

produces n-point functions of deformed QFT [Grosse, L 08] (also used by [Balachandran et. al., ...])

- Moyal space:

$$[X_\mu, X_\nu] = i\theta_{\mu\nu} \neq 0$$

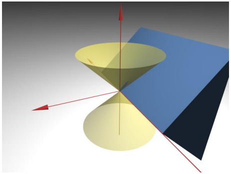
  No pointlike localization possible.

- But there exist lightlike coordinates $X_\pm = X_0 \pm X_2$ such that

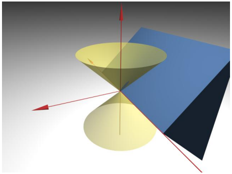
$$[X_+, X_-] = 0$$

  $X_\pm$ can be simultaneously diagonalized.

- Localization in wedges of the form $W = \{x : x_+ > 0\,, x_- < 0\}$ should be possible.
- This can be proven in warped convolution setting.



- U rep. of Poincaré group with positive energy
- $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ v. Neumann alg. "localized in" W



$$W = \{x \in \mathbb{R}^d \,:\, x_1 > |x_0|\}$$

- $U(\lambda)\mathcal{M}U(\lambda)^{-1} \subset \mathcal{M}$ for $\lambda W \subset W$
- $U(\lambda)\mathcal{M}U(\lambda)^{-1} \subset \mathcal{M}'$ for $\lambda W \subset W'$

These conditions are also valid for the deformed algebra

$$\mathcal{M}_\theta := \{A_\theta \,:\, A \in \mathcal{M}^\infty\}''$$

Localization in wedges is compatible with deformation.

Further developments

- Framework applies also to QFT on a class of cosmological spacetimes ("Deformations with Killing fields")
  [Dappiaggi, L, Morfa-Morales 10]
- Discussion of states on Rieffel-deformed C^*-algebra
  (cf. [Kaschek, Neumaier, Waldmann 08])
- Extension to universal deformations for non-abelian groups?
  [Bieliavsky] (κ-deformation, ...)