Construction of quantum field theories with factorizing S-matrices

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Constructive approaches to QFT

- Constructive QFT
  - model/interaction defined by classical Lagrangian
  - quantization, Euclidian techniques
  - rigorous construction of models with polynomial self-interaction in $1+1$ and $2+1$ dim.
- The “form factor program” (FFP)
  - model/interaction defined by a *factorizing S-matrix* (inverse scattering problem)
  - no quantization necessary
  - restricted to $1+1$ dimensions
  - no complete construction achieved yet
- Here: New approach to the construction of models with factorizing S-matrices
  - complementary to FFP
  - with the help of operator-algebraic techniques
  - with the potential of higher-dimensional generalizations

Factorizing S-matrices in two dimensions

- Rapidity parametrization of mass shell, $p(\theta) = m \begin{pmatrix} \cosh\theta \\ \sinh\theta \end{pmatrix}$

Factorizing S-matrices: [Iagolnitzer 78] fixed by *scattering function* S_2,

- for single species of massive particles:

$$(S\Psi^+)_n(\theta_1, \dots, \theta_n) = \prod_{1 \le l < k \le n} S_2(|\theta_l - \theta_k|) \cdot \Psi_n^+(\theta_1, \dots, \theta_n) \,.$$

- factorizing S-matrices known from integrable models (Sin(h)-Gordon, Ising, Thirring [Fröhlich/Seiler 76], ...)

Definition

A **scattering function** is a bounded and continuous function $S_2 : \{\zeta \,:\, 0 \le \mathrm{Im}\,\zeta \le \pi\} \to \mathbb{C}$ which is analytic in the interiour of this strip and satisfies

$$S_2(\theta)^{-1} = \overline{S_2(\theta)} = S_2(-\theta) = S_2(\theta + i\pi)\,, \qquad \theta \in \mathbb{R}\,.$$

Examples: $S_2(\theta) = 1$, $\quad S_2(\theta) = \frac{\sinh\theta - ig}{\sinh\theta + ig}$, $\quad g > 0$

Construction strategy in FFP:

- Consider *form factors* ($A(x)$: local field, Ω: vacuum)

$$F_n^{A(x)}(\theta_1, ..., \theta_n) = \langle \Omega \,|\, A(x) \,|\, \theta_1, ..., \theta_n \rangle_{\text{in}}$$

- 1.) Compute $F_n^{A(x)}$ from consistency eqns *(form factor axioms)*
- 2.) Compute n-point functions

$$\langle A(x)A(y) \rangle \sim \sum_{n=0}^{\infty} \int d^n\boldsymbol{\theta}\, e^{-i\sum_{k=1}^{n} p(\theta_k)\cdot(x-y)}\, |F_n^{A(0)}(\theta_1, ..., \theta_n)|^2$$

  Problem: Convergence not under control

- no complete construction possible in FFP
- Complicated structure of form factors due to locality of $A(x)$
- $\rightarrow$ Construct "weaker" localized fields first, and obtain strictly local observables in a second step

Hilbert space and Zamolodchikov-Faddeev algebra

$$\mathcal{H} := \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$ "S_2-symmetric Fock space" [Liguori/Mintchev 95, GL 03]

- $\mathcal{H}_0 := \mathbb{C} \cdot \Omega$ (Ω: vacuum vector)
- $\mathcal{H}_1 := L^2(\mathbb{R}, d\theta)$ single particle space
- $\mathcal{H}_n \subset L^2(\mathbb{R}^n, d^n\boldsymbol{\theta})$, $n \geq 2$, defined by, $\Psi_n \in \mathcal{H}_n$,

$$\Psi_n(\theta_1, ..., \theta_{k+1}, \theta_k, ..., \theta_n = S_2(\theta_k - \theta_{k+1}) \cdot \Psi_n(\theta_1, ..., \theta_k, \theta_{k+1}, ..., \theta_n)$$

Representation of translations and boosts on $\mathcal{H}$:

$$(U(x,\lambda)\Psi)_n(\theta_1, ..., \theta_n) := \prod_{k=1}^{n} e^{ip(\theta_k)\cdot x} \cdot \Psi_n(\theta_1 - \lambda, ..., \theta_n - \lambda)$$

Total spacetime reflection $j(x) := -x$,

$$(U(0,j)\Psi)_n(\theta_1, ..., \theta_n) := \overline{\Psi_n(\theta_n, ..., \theta_1)}$$

U : (anti)unitary, positive energy, Ω is U-invariant.

- Creation / annihilation operators on $\mathcal{H}$:

$$(z(\theta)\Psi)_n(\theta_1, ..., \theta_n) := \sqrt{n+1}\,\Psi_{n+1}(\theta, \theta_1, ..., \theta_n), \qquad z(\theta)\Omega = 0\,,$$
$$z^\dagger(\theta) := z(\theta)^*$$

- Exchange relations:

$$z(\theta_1)z(\theta_2) = S_2(\theta_1 - \theta_2)\, z(\theta_2)z(\theta_1)\,,$$
$$z^\dagger(\theta_1)z^\dagger(\theta_2) = S_2(\theta_1 - \theta_2)\, z^\dagger(\theta_2)z^\dagger(\theta_1)\,,$$
$$z(\theta_1)z^\dagger(\theta_2) = S_2(\theta_2 - \theta_1)\, z^\dagger(\theta_2)z(\theta_1) + \delta(\theta_1 - \theta_2)\cdot 1$$

Zamolodchikov-Faddeev algebra [Zamolodchikov 79, Faddeev 80]

- Associated quantum field [Schroer 97, Schroer/Wiesbrock 00]

$$\boxed{\phi(x) := \int d\theta\, \left(e^{ip(\theta)\cdot x}\, z^\dagger(\theta) + e^{-ip(\theta)\cdot x}\, z(\theta)\right)}$$

$S_2(\theta) = 1 \Longleftrightarrow \phi$ is the free field

Properties of the field ϕ

For arbitrary $S_2 \neq 1$:

- ϕ is unbounded operator-valued distribution
- $\phi(f) = \int d^2x\, f(x)\phi(x)$ is essentially selfadjoint for real f
- ϕ transforms covariantly under translations and boosts $U(x, \lambda)$
- ϕ creates single particle states from Ω
- ϕ solves Klein-Gordon equation $(\Box + m^2)\phi(x) = 0$
- ϕ does not transform covariantly under the TCP operator $U(0, j)$

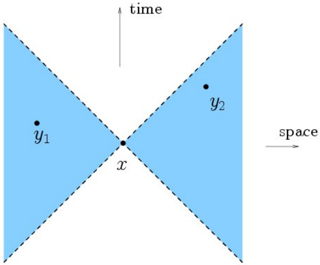
$$\phi'(x) := U(0, j)\phi(-x)U(0, j) \neq \phi(x)$$

- ϕ is not (point-like) local, $[\phi(x), \phi(y)] \neq 0$ for $(x - y)^2 < 0$

Localization properties of ϕ, ϕ'

- Consider $\phi(x)$ and second field $\phi'(x) := U(0,j)\phi(-x)U(0,j)$

$$[\phi(x), \phi'(y_1)] \neq 0$$
$$[\phi(x), \phi'(y_2)] = 0$$

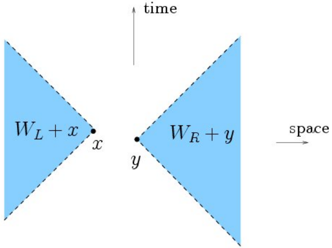


Localization properties of ϕ, ϕ'

- Consider $\phi(x)$ and second field $\phi'(x) := U(0,j)\phi(-x)U(0,j)$

  - $[\phi(x), \phi'(y)] = 0$ if the "wedge" regions $W_L + x$ and $W_R + y$ are spacelike

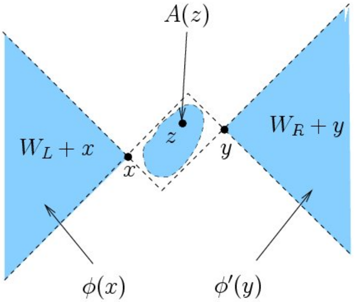
    (use analyticity and crossing of S_2 in proof)



- $\phi(x)$ localized in $W_L + x$ (not at spacetime-point x!)
- ϕ, ϕ' are not the basic physical fields, but auxiliary objects ("polarization-free generators", [Schroer 98, Borchers/Buchholz/Schroer 00])

Local fields/observables

- Wedge-local fields OK, but what about *local* quantum fields/observables?
- Assume $A(x)$ is a local field
  - Locality requires $[\phi(x), A(z)] = 0 = [\phi'(y), A(z)]$
  - Do there exist such local observables? (with the right properties)
  - $\rightarrow$ existence problem for QFT with scattering function S_2



- This question can be answered by operator-algebraic techniques

Operator-algebraic formulation

- ”Wedge algebras“

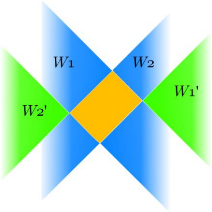
$$\mathcal{A}(W_L + x) := \{e^{i\phi(f)} \;:\; f \in C_0^\infty(W_L + x)\}''$$
$$\mathcal{A}(W_R + y) := \{e^{i\phi'(f)} \;:\; f \in C_0^\infty(W_R + y)\}''$$

Ω is cyclic for each $\mathcal{A}(W)$ (Reeh-Schlieder property)

- Algebras of observables localized in bounded regions, e.g. $\mathcal{O} = W_1 \cap W_2$:

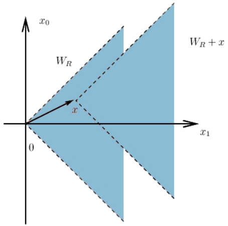
$$\mathcal{A}(W_1 \cap W_2) := \mathcal{A}(W_1) \cap \mathcal{A}(W_2)$$

maximal algebra of observables localized in $W_1 \cap W_2$



- The algebras $\mathcal{A}(\mathcal{O})$ define a local, covariant QFT [Borchers 92]
- **Existence problem:** $\mathcal{A}(\mathcal{O}) \neq \mathbb{C} \cdot 1$? Reeh-Schlieder?

- The algebra $\mathcal{A}(W_R) \cap \mathcal{A}(W_R + x)'$ is related to the inclusion $\mathcal{A}(W_R + x) \subset \mathcal{A}(W_R)$, $x \in W_R$
- $\rightarrow$ study inclusions of von Neumann algebras



- Many techniques available: standard inclusions, split inclusions, classification of factors, Tomita-Takesaki modular theory ...
- Well-suited criterion for $d = 2$: modular nuclearity condition [Buchholz/d'Antoni/Longo 90] Assume that the operator

$$\Xi(x) : \mathcal{A}(W_R) \to \mathcal{H}, \qquad \Xi(x) : A \longmapsto \Delta^{1/4} U(x,1) A\Omega$$

is nuclear (Δ: Modular operator of $(\mathcal{A}(W_R), \Omega)$)

Theorem

If $\Xi(x)$ is nuclear, $x \in W_R$, local observables exist.
More details:

- $\mathcal{A}(W_1) \subset \mathcal{A}(W_2)$ is standard & split if $\overline{W_1} \subset W_2$ [Buchholz/GL 04]
- The Reeh-Schlieder property holds locally, i.e. Ω is cyclic for $\mathcal{A}(W_1 \cap W_2)$ if $(W_1 \cap W_2) \neq \emptyset$ [GL 06]
- Additivity properties [Müger 97, GL 06]

- If modular nuclearity condition holds, a well-defined QFT associated to S_2 exists
- $\rightarrow$ Main task: Prove modular nuclearity condition for suitable S_2
- Helpful fact: Modular group of $(\mathcal{A}(W_R), \Omega)$ coincides with the boosts, $\Delta^{it} = U(0, 2\pi t)$ [Buchholz/GL 04]

- Concrete form of Ξ: Let $A \in \mathcal{A}(W_R)$, $s > 0$, $\Xi(s) := \Xi(\binom{0}{s})$. Then

$$(\Xi(s)A)_n(\theta_1, ..., \theta_n) = \prod_{k=1}^{n} e^{-ms\cosh\theta_k} (A\Omega)_n(\theta_1 - \tfrac{i\pi}{2}, ..., \theta_n - \tfrac{i\pi}{2})$$

- For proof of nuclearity, show that the space of these functions is "almost finite dimensional"
- $\rightarrow$ Study analytic properties of wedge-local wavefunctions $(A\Omega)_n$
- Wedge-locality of A gives holomorphic properties, boundedness of A gives estimates on the analytic continuation
- Need "nice" scattering functions

Definition (regular scattering functions)

A scattering function $S_2 \in \mathcal{S}$ is called regular if there exists $\kappa > 0$ such that S_2 continues to a bounded analytic function on $\mathbb{R} + i(-\kappa, \pi + \kappa)$.

$\mathcal{S}_0 := \{$regular scattering functions$\}$.

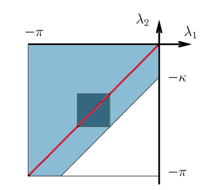
Analyticity domains: $\mathcal{T}_n(\kappa) := \mathbb{R}^n - i(\frac{\pi}{2}, ..., \frac{\pi}{2}) + i(-\kappa, \kappa)^{\times n}$.

Proposition [GL 06]

Let $S_2 \in \mathcal{S}_0$ and $A \in \mathcal{A}(W_R)$.

1. $(A\Omega)_n$ is analytic in $\mathcal{T}_n(\kappa)$.
2. For $0 < \kappa' < \kappa$

$$|(A\Omega)_n(\zeta)| \leq C(S_2, \kappa')^n \cdot \|A\| , \quad \zeta \in \mathcal{T}_n(\kappa') .$$



- This implies that the maps $\Xi_n(s) := P_n\Xi(s)$ are nuclear for $s > 0$. (Use Hardy spaces on tube domains.)
- For nuclearity of $\Xi(s) = \sum_{n=0}^{\infty} \Xi_n(s)$, need good bounds on nuclear norms $\|\Xi_n(s)\|_1$ ($\sum_n \Xi_n(s)$ must converge in $\|\cdot\|_1$-topology)
- $\|\Xi_n(s)\|_1$ are smaller if Pauli principle becomes effective

$$\mathcal{S}_0^{\pm} := \{S_2 \in \mathcal{S}_0 \,:\, S_2(0) = \pm 1\} , \qquad \mathcal{S}_0 = \mathcal{S}_0^+ \cup \mathcal{S}_0^- .$$

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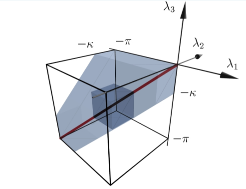

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Proof of the modular nuclearity condition

Theorem [GL 06]

- Let $S_2 \in \mathcal{S}_0$. Then there exists $s_{\min} > 0$ such that $\Xi(s)$ is nuclear for all $s > s_{\min}$.
- Let $S_2 \in \mathcal{S}_0^-$. Then $\Xi(s)$ is nuclear for all $s > 0$.

- For $S_2 \in \mathcal{S}_0^-$, there exist corresponding local QFTs

  (local observables exist, and the Reeh-Schlieder property holds)
- For $S_2 \in \mathcal{S}_0^+$, there might exist a "minimal length"
- All scattering functions known from integrable models are in the class $\mathcal{S}_0^-$
- Construction of models complete in the sense of "algebraic" QFT

Collision states and reconstruction of the S-matrix

- Starting from a scattering function S_2, a QFT was constructed.
- But: Is the S-matrix of this model given by S_2?
  (Inverse scattering problem!)
- Need to compute collision states and the S-matrix of the constructed models

- Scattering theory is possible without explicit knowledge of quantum fields (Haag-Ruelle scattering theory)
- Restriction to $S_2 \in \mathcal{S}_0$ sufficient (localization with arbitrary precision not necessary for scattering theory)

Theorem [GL 06]

Let $S_2 \in \mathcal{S}_0$.

- The constructed model solves the inverse scattering problem for the corresponding S-matrix, i.e. its scattering operator is

$$(S\Psi^+)_n(\theta_1, ..., \theta_n) = \prod_{1 \leq l < k \leq n} S_2(|\theta_l - \theta_k|) \cdot \Psi_n^+(\theta_1, ..., \theta_n) \, .$$

- Idealized n-particle collision states are given by

$$z^\dagger(\theta_1) \cdots z^\dagger(\theta_n)\Omega = |\, \theta_1, ..., \theta_n\rangle_{\rm out} \, , \qquad \theta_1 < ... < \theta_n \, ,$$
$$z^\dagger(\theta_1) \cdots z^\dagger(\theta_n)\Omega = |\, \theta_1, ..., \theta_n\rangle_{\rm in} \, , \qquad \theta_1 > ... > \theta_n \, ,$$

- Asymptotic completeness holds.

$\rightarrow$ Rigorous justification of the motivation for Zamolodchikov's algebra.

Summary

- New approach to constructive QFT:
  - Construct wedge-local quantum fields first (Easier because of weak localization)
  - Determine algebras of local observables by commutation relations with wedge-local fields
- Explicit realization of this program for special models in $d = 2$
  - Starting from factorizing S-matrix, construct wedge-local quantum fields ϕ, ϕ' explicitly (Zamolodchikov's algebra)
  - Prove existence of local observables via split property and modular nuclearity condition
  - Information about the local structure can be extracted from estimates on wedge-local quantities!
- Construction works for large class of factorizing S-matrices
  - Inverse scattering problem solved
  - Proof of asymptotic completeness
  - Many new models found