Construction of quantum field theories with factorizing S-matrices

Gandalf Lechner

University of Vienna

École Polytechnique

14.03.2008
Constructive approaches to QFT

- **Constructive QFT**
  - model/interaction defined by classical Lagrangian
  - quantization, Euclidian techniques
  - rigorous construction of models with polynomial self-interaction in 1 + 1 and 2 + 1 dim.

- **The “form factor program” (FFP)**
  - model/interaction defined by a *factorizing S-matrix* (inverse scattering problem)
  - no quantization necessary
  - restricted to 1 + 1 dimensions
  - no complete construction achieved yet

- **Here**: New approach to the construction of models with factorizing S-matrices
  - complementary to FFP
  - with the help of operator-algebraic techniques
  - with the potential of higher-dimensional generalizations
Factorizing S-matrices in two dimensions

- Rapidity parametrization of mass shell, \( p(\theta) = m \left( \frac{\cosh \theta}{\sinh \theta} \right) \)

**Factorizing S-matrices:** [Iagolnitzer 78] fixed by **scattering function** \( S_2 \),
- for single species of massive particles:

\[
(S\Psi^+)_n(\theta_1, \ldots, \theta_n) = \prod_{1 \leq l < k \leq n} S_2(|\theta_l - \theta_k|) \cdot \Psi^+_n(\theta_1, \ldots, \theta_n).
\]

- factorizing S-matrices known from integrable models
  (Sin(h)-Gordon, Ising, Thirring [Fröhlich/Seiler 76], ...)

**Definition**

A **scattering function** is a bounded and continuous function \( S_2 : \{\zeta : 0 \leq \text{Im} \zeta \leq \pi\} \rightarrow \mathbb{C} \) which is analytic in the interior of this strip and satisfies

\[
S_2(\theta)^{-1} = \overline{S_2(\theta)} = S_2(-\theta) = S_2(\theta + i\pi), \quad \theta \in \mathbb{R}.
\]

Examples: \( S_2(\theta) = 1 \), \( S_2(\theta) = \frac{\sinh \theta - ig}{\sinh \theta + ig}, \ g > 0 \)
Construction strategy in FFP:

- Consider *form factors* \((A(x): \text{local field}, \Omega: \text{vacuum})\)

\[
F_n^A(x)(\theta_1,\ldots,\theta_n) = \langle \Omega | A(x) | \theta_1,\ldots,\theta_n \rangle_{\text{in}}
\]

1.) Compute \(F_n^A(x)\) from consistency eqns (*form factor axioms*)

2.) Compute \(n\)-point functions

\[
\langle A(x)A(y) \rangle \sim \sum_{n=0}^{\infty} \int d^m \theta \, e^{-i \sum_{k=1}^{n} p(\theta_k) \cdot (x-y)} |F_n^A(0)(\theta_1,\ldots,\theta_n)|^2
\]

**Problem:** Convergence not under control

- no complete construction possible in FFP

- Complicated structure of form factors due to locality of \(A(x)\)

→ Construct “weaker” localized fields first, and obtain strictly local observables in a second step
Hilbert space and Zamolodchikov-Faddeev algebra

\[ \mathcal{H} := \bigoplus_{n=0}^{\infty} \mathcal{H}_n \quad \text{“} S_2\text{-symmetric Fock space”} \quad \text{[Liguori/Mintchev 95, GL 03]} \]

- \( \mathcal{H}_0 := \mathbb{C} \cdot \Omega \) \quad (\Omega: \text{vacuum vector})
- \( \mathcal{H}_1 := L^2(\mathbb{R}, d\theta) \) \quad \text{single particle space}
- \( \mathcal{H}_n \subset L^2(\mathbb{R}^n, d^n\theta), \, n \geq 2 \), defined by, \( \Psi_n \in \mathcal{H}_n \),

\[ \Psi_n(\theta_1, ..., \theta_{k+1}, \theta_k, ..., \theta_n) = S_2(\theta_k - \theta_{k+1}) \cdot \Psi_n(\theta_1, ..., \theta_k, \theta_{k+1}, ..., \theta_n) \]

Representation of translations and boosts on \( \mathcal{H} \):

\[ (U(x, \lambda)\Psi)_n(\theta_1, ..., \theta_n) := \prod_{k=1}^{n} e^{ip(\theta_k) \cdot x} \cdot \Psi_n(\theta_1 - \lambda, ..., \theta_n - \lambda) \]

Total spacetime reflection \( j(x) := -x \),

\[ (U(0, j)\Psi)_n(\theta_1, ..., \theta_n) := \overline{\Psi_n(\theta_n, ..., \theta_1)} \]

\( U \) : (anti)unitary, positive energy, \( \Omega \) is \( U \)-invariant.
Creation / annihilation operators on $\mathcal{H}$:

$$(z(\theta)\Psi)_n(\theta_1, \ldots, \theta_n) := \sqrt{n + 1} \Psi_{n+1}(\theta, \theta_1, \ldots, \theta_n), \quad z(\theta)\Omega = 0,$$

$$z^\dagger(\theta) := z(\theta)^*$$

Exchange relations:

$$z(\theta_1)z(\theta_2) = S_2(\theta_1 - \theta_2) z(\theta_2)z(\theta_1),$$

$$z^\dagger(\theta_1)z^\dagger(\theta_2) = S_2(\theta_1 - \theta_2) z^\dagger(\theta_2)z^\dagger(\theta_1),$$

$$z(\theta_1)z^\dagger(\theta_2) = S_2(\theta_2 - \theta_1) z^\dagger(\theta_2)z(\theta_1) + \delta(\theta_1 - \theta_2) \cdot 1$$

Zamolodchikov-Faddeev algebra [Zamolodchikov 79, Faddeev 80]

Associated quantum field [Schroer 97, Schroer/Wiesbrock 00]

$$\phi(x) := \int d\theta \left( e^{ip(\theta) \cdot x} z^\dagger(\theta) + e^{-ip(\theta) \cdot x} z(\theta) \right)$$

$S_2(\theta) = 1 \iff \phi$ is the free field
Properties of the field $\phi$

For arbitrary $S_2 \neq 1$:
- $\phi$ is unbounded operator-valued distribution
- $\phi(f) = \int d^2x \, f(x)\phi(x)$ is essentially selfadjoint for real $f$
- $\phi$ transforms covariantly under translations and boosts $U(x, \lambda)$
- $\phi$ creates single particle states from $\Omega$
- $\phi$ solves Klein-Gordon equation $(\Box + m^2)\phi(x) = 0$
- $\phi$ does not transform covariantly under the TCP operator $U(0, j)$
  
  $$\phi'(x) := U(0, j)\phi(-x)U(0, j) \neq \phi(x)$$
- $\phi$ is not (point-like) local, $[\phi(x), \phi(y)] \neq 0$ for $(x - y)^2 < 0$
Consider $\phi(x)$ and second field $\phi'(x) := U(0, j)\phi(-x)U(0, j)$

\[
\begin{align*}
[\phi(x), \phi'(y_1)] &\neq 0 \\
[\phi(x), \phi'(y_2)] &= 0
\end{align*}
\]
Localization properties of $\phi, \phi'$

- Consider $\phi(x)$ and second field $\phi'(x) := U(0,j)\phi(-x)U(0,j)$

- $[\phi(x), \phi'(y)] = 0$ if the “wedge” regions $W_L + x$ and $W_R + y$ are spacelike

  (use analyticity and crossing of $S_2$ in proof)

- $\phi(x)$ localized in $W_L + x$ (not at spacetime-point $x$!)

- $\phi, \phi'$ are not the basic physical fields, but auxiliary objects (“polarization-free generators”, [Schroer 98, Borchers/Buchholz/Schroer 00])

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{wedge_regions}
\caption{Wedge regions $W_L + x$ and $W_R + y$}
\end{figure}
Local fields/observables

- Wedge-local fields OK, but what about *local* quantum fields/observables?
- Assume $A(x)$ is a local field

  - Locality requires
    $$[\phi(x), A(z)] = 0 = [\phi'(y), A(z)]$$
  - Do there exist such local observables? (with the right properties)
  - $\rightarrow$ existence problem for QFT with scattering function $S_2$

- This question can be answered by operator-algebraic techniques
"Wedge algebras"

\[
\mathcal{A}(W_L + x) := \{ e^{i\phi(f)} : f \in C_0^\infty(W_L + x) \}''
\]

\[
\mathcal{A}(W_R + y) := \{ e^{i\phi'(f)} : f \in C_0^\infty(W_R + y) \}''
\]

\(\Omega\) is cyclic for each \(\mathcal{A}(W)\) (Reeh-Schlieder property)

Algebras of observables localized in bounded regions, e.g.
\(\mathcal{O} = W_1 \cap W_2:\)

\[
\mathcal{A}(W_1 \cap W_2) := \mathcal{A}(W_1) \cap \mathcal{A}(W_2)
\]

maximal algebra of observables localized in \(W_1 \cap W_2\)

The algebras \(\mathcal{A}(\mathcal{O})\) define a local, covariant QFT [Borchers 92]

Existence problem: \(\mathcal{A}(\mathcal{O}) \neq \mathbb{C} \cdot 1\) ? Reeh-Schlieder?
Wedge inclusions & Existence of local observables

- The algebra $\mathcal{A}(W_R) \cap \mathcal{A}(W_R + x)'$ is related to the inclusion $\mathcal{A}(W_R + x) \subset \mathcal{A}(W_R)$, $x \in W_R$

- → study inclusions of von Neumann algebras

- Many techniques available: standard inclusions, split inclusions, classification of factors, Tomita-Takesaki modular theory ...

- Well-suited criterion for $d = 2$: modular nuclearity condition

[Buchholz/d’Antoni/Longo 90] Assume that the operator $\Xi(x) : \mathcal{A}(W_R) \to \mathcal{H}$, $\Xi(x) : A \mapsto \Delta^{1/4} U(x, 1) A \Omega$

is nuclear ($\Delta$: Modular operator of $(\mathcal{A}(W_R), \Omega)$)
Theorem

If $\Xi(x)$ is nuclear, $x \in W_R$, local observables exist.

More details:

- $A(W_1) \subset A(W_2)$ is standard & split if $\overline{W_1} \subset W_2$ [Buchholz/GL 04]
- The Reeh-Schlieder property holds locally, i.e. $\Omega$ is cyclic for $A(W_1 \cap W_2)$ if $(W_1 \cap W_2) \neq \emptyset$ [GL 06]
- Additivity properties [Müger 97, GL 06]

- If modular nuclearity condition holds, a well-defined QFT associated to $S_2$ exists
- $\rightarrow$ Main task: Prove modular nuclearity condition for suitable $S_2$
- Helpful fact: Modular group of $(A(W_R), \Omega)$ coincides with the boosts, $\Delta^{it} = U(0, 2\pi t)$ [Buchholz/GL 04]
Concrete form of $\Xi$: Let $A \in \mathcal{A}(W_R)$, $s > 0$, $\Xi(s) := \Xi(\binom{0}{s})$. Then

$$(\Xi(s)A)_n(\theta_1, \ldots, \theta_n) = \prod_{k=1}^{n} e^{-ms \cosh(\theta_k)} (A\Omega)_n(\theta_1 - \frac{i\pi}{2}, \ldots, \theta_n - \frac{i\pi}{2})$$

For proof of nuclearity, show that the space of these functions is “almost finite dimensional”

$\rightarrow$ Study analytic properties of wedge-local wavefunctions $(A\Omega)_n$

Wedge-locality of $A$ gives holomorphic properties, boundedness of $A$ gives estimates on the analytic continuation

Need “nice” scattering functions

**Definition (regular scattering functions)**

A scattering function $S_2 \in \mathcal{S}$ is called regular if there exists $\kappa > 0$ such that $S_2$ continues to a bounded analytic function on $\mathbb{R} + i(-\kappa, \pi + \kappa)$.

$S_0 := \{ \text{regular scattering functions} \}$. 
Analyticity domains: \( \mathcal{T}_n(\kappa) := \mathbb{R}^n - i(\frac{\pi}{2}, \ldots, \frac{\pi}{2}) + i(-\kappa, \kappa)^{\times n} \).

**Proposition [GL 06]**

Let \( S_2 \in S_0 \) and \( A \in \mathcal{A}(W_R) \).

1. \((A\Omega)_n\) is analytic in \( \mathcal{T}_n(\kappa) \).
2. For \( 0 < \kappa' < \kappa \)

\[
|(A\Omega)_n(\zeta)| \leq C(S_2, \kappa')^n \cdot \|A\|, \quad \zeta \in \mathcal{T}_n(\kappa').
\]

This implies that the maps \( \Xi_n(s) := P_n \Xi(s) \) are nuclear for \( s > 0 \).

(Use Hardy spaces on tube domains.)

For nuclearity of \( \Xi(s) = \sum_{n=0}^{\infty} \Xi_n(s) \), need good bounds on nuclear norms \( \|\Xi_n(s)\|_1 \) (\( \sum_{n} \Xi_n(s) \) must converge in \( \| \cdot \|_1 \)-topology)

\( \|\Xi_n(s)\|_1 \) are smaller if Pauli principle becomes effective

\[
S_0^\pm := \{ S_2 \in S_0 : S_2(0) = \pm 1 \}, \quad S_0 = S_0^+ \cup S_0^-.
\]
Analyticity domains: \( \mathcal{T}_n(\kappa) := \mathbb{R}^n - i\left(\frac{\pi}{2}, \ldots, \frac{\pi}{2}\right) + i(-\kappa, \kappa)^\times n. \)

**Proposition [GL 06]**

Let \( S_2 \in S_0 \) and \( A \in \mathcal{A}(W_R). \)

1. \((A\Omega)_n\) is analytic in \( \mathcal{T}_n(\kappa) \).

2. For \( 0 < \kappa' < \kappa \)
   \[
   |(A\Omega)_n(\zeta)| \leq C(S_2, \kappa')^n \cdot \|A\|, \quad \zeta \in \mathcal{T}_n(\kappa').
   \]

- This implies that the maps \( \Xi_n(s) := P_n\Xi(s) \) are nuclear for \( s > 0 \).
  (Use Hardy spaces on tube domains.)

- For nuclearity of \( \Xi(s) = \sum_{n=0}^{\infty} \Xi_n(s) \), need good bounds on nuclear norms \( \|\Xi_n(s)\|_1 \) \( (\sum_n \Xi_n(s) \) must converge in \( \| \cdot \|_1\)-topology)

- \( \|\Xi_n(s)\|_1 \) are smaller if Pauli principle becomes effective

\[
S_0^\pm := \{S_2 \in S_0 : S_2(0) = \pm 1\}, \quad S_0 = S_0^+ \cup S_0^-.
\]
Proof of the modular nuclearity condition

**Theorem [GL 06]**

- Let $S_2 \in S_0$. Then there exists $s_{\text{min}} > 0$ such that $\Xi(s)$ is nuclear for all $s > s_{\text{min}}$.
- Let $S_2 \in S_0^-$. Then $\Xi(s)$ is nuclear for all $s > 0$.

- For $S_2 \in S_0^-$, there exist corresponding local QFTs
  (local observables exist, and the Reeh-Schlieder property holds)
- For $S_2 \in S_0^+$, there might exist a “minimal length”
- All scattering functions known from integrable models are in the class $S_0^-$
- Construction of models complete in the sense of “algebraic” QFT
• Starting from a scattering function $S_2$, a QFT was constructed.
• **But**: Is the S-matrix of this model given by $S_2$?
  (Inverse scattering problem!)
• Need to compute collision states and the S-matrix of the constructed models

• Scattering theory is possible without explicit knowledge of quantum fields (Haag-Ruelle scattering theory)
• Restriction to $S_2 \in S_0$ sufficient (localization with arbitrary precision not necessary for scattering theory)
Theorem [GL 06]

Let \( S_2 \in S_0 \).

- The constructed model solves the inverse scattering problem for the corresponding S-matrix, i.e. its scattering operator is

\[
(S\Psi^+)_n(\theta_1, \ldots, \theta_n) = \prod_{1 \leq l < k \leq n} S_2(|\theta_l - \theta_k|) \cdot \Psi_n^+(\theta_1, \ldots, \theta_n).
\]

- Idealized \( n \)-particle collision states are given by

\[
\begin{align*}
\hat{z}^\dagger(\theta_1) \cdots \hat{z}^\dagger(\theta_n)\Omega &= |\theta_1, \ldots, \theta_n\rangle_{\text{out}}, & \theta_1 < \ldots < \theta_n, \\
\hat{z}^\dagger(\theta_1) \cdots \hat{z}^\dagger(\theta_n)\Omega &= |\theta_1, \ldots, \theta_n\rangle_{\text{in}}, & \theta_1 > \ldots > \theta_n,
\end{align*}
\]

- Asymptotic completeness holds.

→ Rigorous justification of the motivation for Zamolodchikov’s algebra.
New approach to constructive QFT:
- Construct wedge-local quantum fields first (Easier because of weak localization)
- Determine algebras of local observables by commutation relations with wedge-local fields

Explicit realization of this program for special models in $d = 2$
- Starting from factorizing S-matrix, construct wedge-local quantum fields $\phi, \phi'$ explicitly (Zamolodchikov’s algebra)
- Prove existence of local observables via split property and modular nuclearity condition
- Information about the local structure can be extracted from estimates on wedge-local quantities!

Construction works for large class of factorizing S-matrices
- Inverse scattering problem solved
- Proof of asymptotic completeness
- Many new models found