Modular Localization and Constructive Algebraic QFT

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This talk was the second in a series of three (by R. Longo, myself, and Y. Tanimoto, respectively) on applications of Tomita-Takesaki modular theory in algebraic quantum field theory. The focus of my talk was the construction of examples of algebraic quantum field theories.

A central notion in this context is that of a *Borchers triple*. In the simplest two-dimensional situation, this consists of a von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathcal{H}$, a unitary strongly continuous positive energy representation $U(x_+, x_-) = e^{ix_+P_+}e^{ix_-P_-}$ of $\mathbb{R}^2$ such that $\mathrm{ad}U(x_+, x_-)(\mathcal{M}) \subset \mathcal{M}$ for $x_+ \geq 0$, $x_- \leq 0$, and a unit vector $\Omega \in \mathcal{H}$ that is invariant under U and cyclic and separating for $\mathcal{M}$ (see [2], and [5] for a review).

Given a Borchers triple $(\mathcal{M}, U, \Omega)$, the representation U extends from $\mathbb{R}^2$ to the proper Poincaré group $\mathcal{P}$ by a theorem of Borchers [1]. The algebra $\mathcal{M}$ can be interpreted as being localized in the wedge region $W = \{x \in \mathbb{R}^2 : x_+ > 0, x_- < 0\}$, and by a canonical procedure, the triple defines a map from open subsets $\mathcal{O} \subset \mathbb{R}^2$ to von Neumann algebras $\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ that is inclusion-preserving, local, covariant under the representation U, and fixed by $\mathcal{A}(W) = \mathcal{M}$.

If this map has also the property that Ω is cyclic for $\mathcal{A}(\mathcal{O})$ for every non-empty $\mathcal{O}$, these data describe a quantum field theory in its vacuum representation. One is therefore interested in finding examples of Borchers triples. Free field theory examples of Borchers triples are well-known.

In this talk, I reviewed two procedures for constructing examples of Borchers triples. The first [2] is related to Rieffel deformations of C^*-algebras [7] and is based on the notion of a *warped convolution*, a deformation procedure for operators in $\mathcal{B}(\mathcal{H})$. Let $(\mathcal{M}, U, \Omega)$ be a Borchers triple (for example, one given by a free field theory), and let $A \in \mathcal{M}$ be an operator that is smooth w.r.t. U, i.e. such that $x \mapsto \mathrm{ad}(U(x))(A)$ is smooth in norm. As a deformation parameter, consider a 2×2 matrix Q antisymmetric w.r.t. the Minkowski inner product. Then the warped convolution of A is defined as

$$A_Q = (2\pi)^{-2} \int_{\mathbb{R}^2} dp \int_{\mathbb{R}^2} dx\, e^{-ip\cdot x}\, \mathrm{ad}(U(Qp))(A)\, U(x).$$

This integral exists in an oscillatory sense on the dense domain of smooth vectors in $\mathcal{H}$ and extends to a bounded operator on all of $\mathcal{H}$. Denote the von Neumann algebra generated by all A_Q, $A \in \mathcal{M}$ smooth, by $\mathcal{M}_Q$.

Then the main theorem in this context, a consequence of various properties of the map $A \mapsto A_Q$, is that the triple $(\mathcal{M}_Q, U, \Omega)$ is again a Borchers triple if Q satisfies a positivity condition related to the spectrum of U. One thus obtains a family of Borchers triples, indexed by Q, from a given one. Triples with different parameters are inequivalent, but they all have the special property that the modular data of $\mathcal{M}_Q$ are independent of Q, i.e. coincide with the modular data of the original von Neumann algebra $\mathcal{M}_0 = \mathcal{M}$. It is an open problem whether the "local" von Neumann algebras $\mathcal{A}_Q(\mathcal{O})$ generated from $(\mathcal{M}_Q, U, \Omega)$ are non-trivial or not.

The second procedure for constructing Borchers triples is based on the notion of a crossing-symmetric R-matrix. One starts from the single particle Hilbert space $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta) \otimes \mathcal{K}$, where $\mathcal{K}$ is a separable (often times finite-dimensional) Hilbert space for internal degrees of freedom, and $L^2(\mathbb{R}, d\theta)$ carries the usual realization of the unitary massive irreducible positive energy representation U_1 of $\mathcal{P}$. In this context, a crossing-symmetric R-matrix is a function R from $\mathbb{R}$ to unitaries on $\mathcal{K} \otimes \mathcal{K}$ satisfying a number of properties. In particular, $R(\theta)$ is required to satisfy the Yang-Baxter equation with spectral parameter θ and a symmetry condition that ensures that R generates a unitary representation of the symmetric group S_n on $\mathcal{H}_1^{\otimes n}$. Denoting by $\mathcal{H}_n \subset \mathcal{H}_1^{\otimes n}$ the subspace on which this representation acts trivially, the Hilbert space of the Borchers triple to be constructed is then $\mathcal{H} := \bigoplus_n \mathcal{H}_n$. The representation U is defined by second quantization of U_1, and Ω is defined as the Fock vacuum of $\mathcal{H}$.

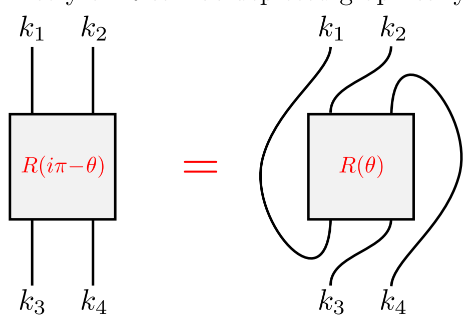
Similar to a usual Bose Fock space, also $\mathcal{H}$ carries canonical creation / annihilation operators, and their sum defines a quantum field ϕ [6]. These field operators generate a von Neumann algebra $\mathcal{M}$ such that $(\mathcal{M}, U, \Omega)$ is a Borchers triple if R extends to a bounded analytic function on the complex strip $0 < \mathrm{Im}\theta < \pi$ and the crossing symmetry

$$\langle k_1 \otimes k_2,\, R(i\pi - \theta)\, k_3 \otimes k_4 \rangle = \langle k_2 \otimes \Gamma k_4,\, R(\theta)\, \Gamma k_1 \otimes k_3 \rangle, \qquad \theta \in \mathbb{R},$$

reminiscent of the KMS condition. Here $k_1, \ldots, k_4$ are arbitrary vectors in $\mathcal{K}$, and Γ is an antiunitary involution on $\mathcal{K}$ related to the modular conjugation of $\mathcal{M}$ (see [3]). The local algebras are non-trivial if R satisfies further regularity conditions.

To make connections to other talks given at this meeting, let us make the following two observations.

(1) The crossing symmetry of R can be depicted graphically as



which suggests a relation to the string Fourier transform presented in the talk by A. Jaffe.

(2) The domain of the one-particle component of the modular operator $\Delta^{1/2}$ becomes a complex Hilbert space when closed in its graph norm. For

the setting of the second example, this Hilbert space coincides with the classical Hardy space on the strip $0 < \text{Im}\theta < \pi$ [4], and thus connects to the reproducing kernel Hilbert space setting discussed by P. Jorgensen.

References

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